Matrix combination of BFKL and DGLAP

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LPTHE, Universities of Paris VI and VII and CNRS

Work with M. Ciafaloni, D. Colferai and A. Stasto [arXiv:0707.1453]
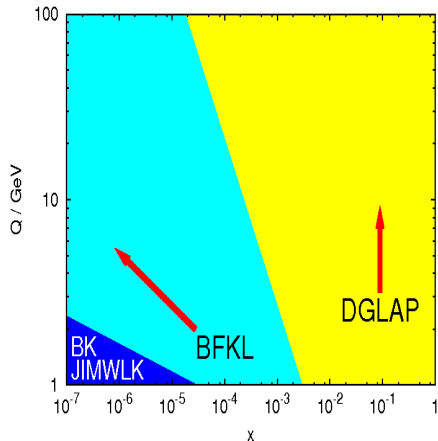
ISMD 2007
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This talk is a progress report on a long-term project to put together *DGLAP* and the linear regime of *BFKL* evolution, including higher order and running-coupling corrections.

Main groups active:

▶ Altarelli, Ball, Forte (+ Falgari, Marzano)   aka ABF
▶ Ciafaloni, Colferai, GPS, Staśto   aka CCSS
+ Thorne & White

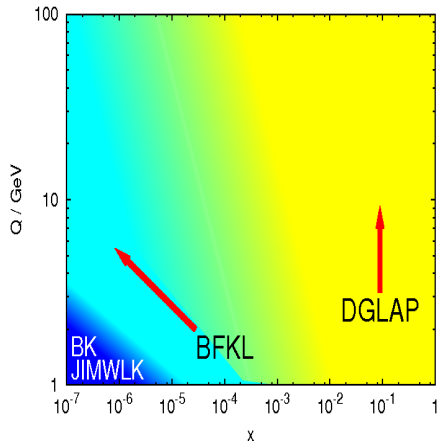When looking at proton structure we can establish different evolution regimes:

But:

▶ Regions of validity not clearly delimited

▶ Higher orders of DGLAP contaminated by leading BFKL:

$$P_{gg}(x) \simeq \frac{\bar{\alpha}_s}{x} + \bar{\alpha}_s^4 \frac{\zeta(3)}{3} \frac{\ln^3 x}{x} + \dots$$

▶ Higher orders of BFKL contaminated by leading DGLAP:

$$K(k, k') \simeq \bar{\alpha}_s - \bar{\alpha}_s^2 \frac{11}{12} \ln \frac{k^2}{k'^2} + \dots$$

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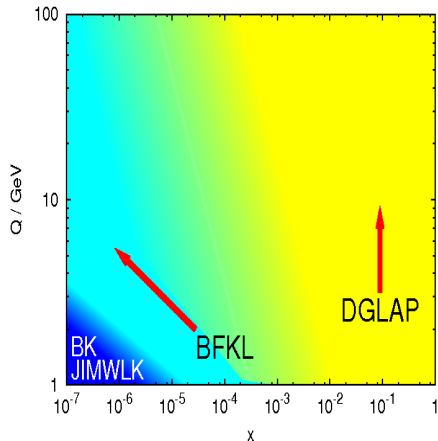

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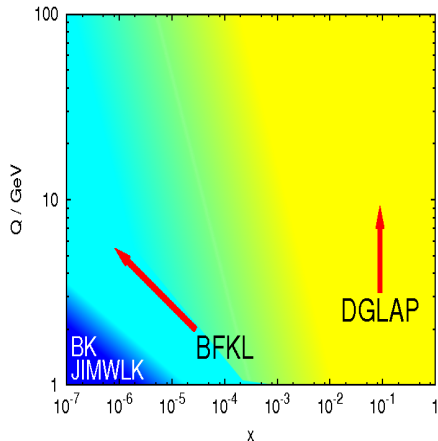

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DGLAP, BFKL (fixed coupling)

DGLAP

Integro(x)–differential(Q^2) eq^n for *integrated* gluon dist., g:

$$\frac{dg(x, Q^2)}{d \ln Q^2} =$$

$$\int \frac{dz}{z} P_{gg}(z) g(\frac{x}{z}, Q^2)$$

BFKL

Integro(k)–differential(x) eq^n for *unintegrated* gluon dist., G:

$$\frac{dG(x, k^2)}{d \ln 1/x} =$$

$$\int \frac{dk'^2}{k'^2} K(k/k') G(x, k'^2)$$

k, Q are transverse scales; x is longitudinal mom. fraction

$$xg(x, Q^2) = \int^Q d^2k\, G(x, k^2)$$

Both DGLAP and BFKL relate $\perp$ structure to long. structure:

▶ given long. struct. DGLAP gives you $\perp$ struct. evolution

▶ given $\perp$ struct. BFKL gives you long. struct. evolution

When calculated at all orders they must encode the same physics.

Inevitable that one contaminated by other at fixed order

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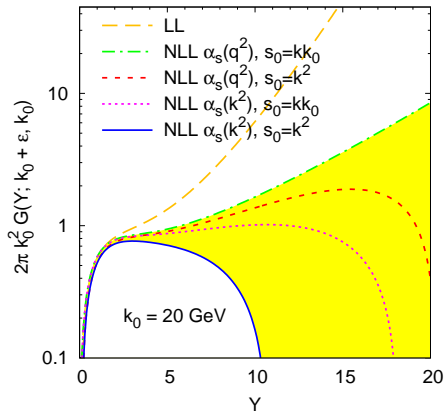

Choices that formally only affect NNLLx:

▶ scale of α_s

▶ 'energy-scale' s_0 ($Y = \ln s/s_0$).

lead to completely different answers

Source of instability is presence in NLL BFKL of a truncated subset of DGLAP. Only way to get stability is to include full DGLAP.

NLL Green function solution

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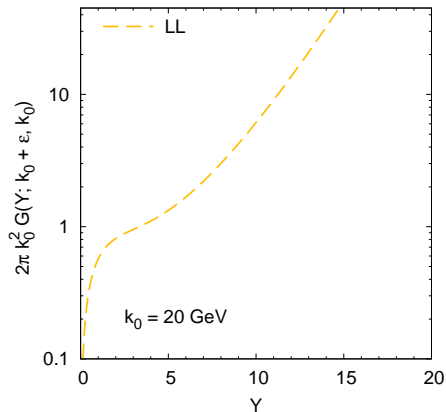
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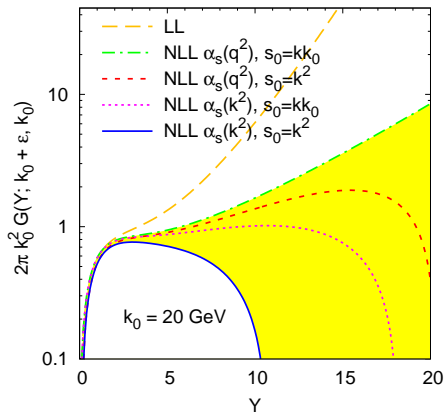

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Merging BFKL & DGLAP

Long history of work on merging leading BFKL and DGLAP.

CCFM '88; Lund group $\sim$ '95; Durham-Cracow group $\sim$'95;

Two approaches have been used in order to combine BFKL and DGLAP *including higher orders:*

▶ Establish all-order relation (*duality relation*) between splitting functions (DGLAP) and evolution kernel (BFKL). Use that to simultaneously construct splitting functions consistent with BFKL kernel and vice-versa.

Altarelli, Ball & Forte '99–

▶ Establish a more general equation that embodies both BFKL and DGLAP (*double-integral equation*):

$$G(x, k^2) = G_0(x, k^2) + \int dz \int dk'^2 \frac{dk'^2}{k'^2} K(z, k, k') G(x/z, k'^2)$$

From that, deduce *effective* splitting function and BFKL kernel.

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Write Kernel as power series in α_s: $K = \sum_{n=0} \hat{\alpha}^n K_n$ $\hat{\alpha} = \alpha_s/2\pi$

First order (*LLx-LO*) has two parts:

$$K_0(\gamma, \omega) = \underbrace{\frac{2C_A}{\omega} \chi_0^\omega(\gamma)}_{\text{BFKL (LLx)}} + \underbrace{\left[\Gamma_{gg,0}(\omega) - \frac{2C_A}{\omega} \right] \chi_c^\omega(\gamma)}_{\text{finite-}x\text{ DGLAP (LO)}}$$

use Mellin transforms: $\gamma \leftrightarrow k^2$, $\omega \leftrightarrow \ln 1/x$, $\Gamma_{gg,0}(\omega) \leftrightarrow P_{gg}(x)$

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BFKL piece has usual transverse structure with *kinematic constraint*

$$\chi_0^\omega(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 + \omega - \gamma)$$

Note symmetry $\gamma \leftrightarrow 1 - \gamma + \omega$

Multiplied by $\alpha_s(q^2)$, $\vec{q} = \vec{k} - \vec{k}'$

DGLAP remainder piece has a *collinear kernel*:

$$\chi_c^\omega(\gamma) = \frac{1}{\gamma} + \frac{1}{1 + \omega - \gamma}$$

Multiplied by $\alpha_s(k_>^2)$

Write Kernel as power series in α_s: $K = \sum_{n=0} \hat{\alpha}^n K_n$ $\qquad \hat{\alpha} = \alpha_s/2\pi$
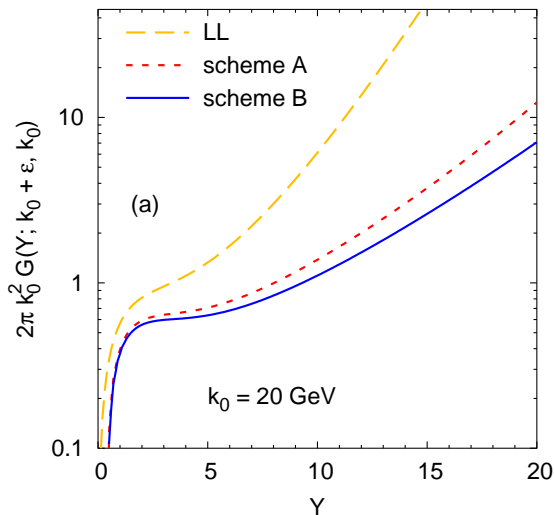
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Next order (*NLx-NLO*) also has two parts:

$$K_1(\gamma, \omega) = \frac{(2C_A)^2}{\omega} \tilde{\chi}_1^\omega(\gamma) + \tilde{\Gamma}_{gg,1}(\omega) \chi_c^\omega(\gamma)$$

with $\tilde{\chi}_1$ and $\tilde{\Gamma}_{gg,1}(\omega)$ adjusted so as to reproduce NLx BFKL and NLO DGLAP.

Green fn. from improved kernel

(a)

k_0 = 20 GeV

LL
scheme A
scheme B

$2\pi k_0^2 G(Y; k_0 + \epsilon, k_0)$

Y

First tried in '03, without NLO DGLAP piece.

NLx-LO

Two schemes, to estimate degree of stability

▶ scheme A violates mom. sum-rule at $\mathcal{O}\left(\alpha_s^2\right)$

▶ scheme B satisfies it at all orders

Solve double-integral eqⁿ with each.

Different schemes → similar results

Green function $\Rightarrow$ effective P_{gg}

Construct a gluon density from Green function (take $k \gg k_0$):

$$xg(x, Q^2) \equiv \int^Q d^2k \; G^{(\nu_0 = k^2)}(\ln 1/x, k, k_0)$$

Numerically solve equation for effective splitting function, $P_{gg,\text{eff}}(z, Q^2)$:

$$\frac{dg(x, Q^2)}{d \ln Q^2} = \int \frac{dz}{z} P_{gg,\text{eff}}(z, Q^2) \, g\left(\frac{x}{z}, Q^2\right)$$

Factorisation

▶ Splitting function:

  red paths

▶ Green function:

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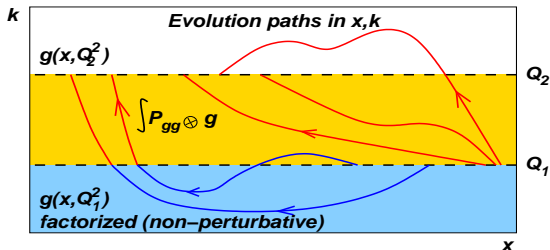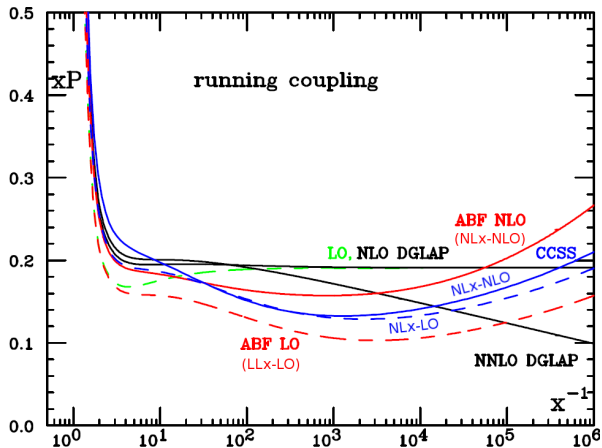Construct a gluon density from Green function (take $k \gg k_0$):

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One channel P_{gg}: ABF v. CCSS

Altarelli, Ball & Forte have also calculated effective P_{gg}:

- ▶ similar physical ingredients
- ▶ completely different 'implementation'

Main features similar between CCSS & ABF.

In particular splitting-fn has **dip** at $x \sim 10^{-3}$.

Two channels

BFKL is naturally single-channel   Only gluon production has $1/x$ divergence

DGLAP is multi-channel   Quarks and gluons both have collinear divergences

So far we had *ignored the multi-channel aspect,* for simplicity. But:

▶ If we are to use small-x resummed splitting functions, we need the whole singlet matrix

▶ Including quarks in evolution may provide a convenient way of resumming collinear logs in impact factors

Generalise double-integral eq^n to two channels

Add flavour indices to Green function and kernel

$$G_{ab}(x, k^2, k_0^2) = \delta^2(k - k_0)\delta_{ab} + \int dz \int dk'^2 \frac{dk'^2}{k'^2} K_{ac}(z, k, k') G_{cb}(x/z, k'^2, k_0^2)$$

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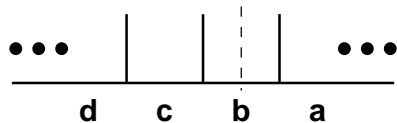DGLAP is multi-channel   Quarks and gluons both have collinear divergences

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Want to encode two strongly ordered collinear limits

DGLAP limit　　$\dots \Gamma_{dc}\,\Gamma_{cb}\,\Gamma_{ba}\,\dots$

$$\begin{cases} x_d < x_c < x_b < x_a \\ k_{td} \gg k_{tc} \gg k_{tb} \gg k_{ta} \end{cases}$$

anti-DGLAP limit　$\dots \Gamma_{cd}\,\Gamma_{bc}\,\Gamma_{ab}\,\dots$

$$\begin{cases} \bar{x}_d > \bar{x}_c > \bar{x}_b > \bar{x}_a \\ k_{td} \ll k_{tc} \ll k_{tb} \ll k_{ta} \end{cases}$$

$$= \dots (\Gamma^T)_{dc}\,(\Gamma^T)_{cb}\,(\Gamma^T)_{ba}\,\dots\,.$$

Suggests sym. $K(\gamma,\omega) = K^T(1+\omega-\gamma,\omega)$. But this $\rightarrow$ spurious colour & $1/\omega$ structures, e.g. $\alpha_s^2 C_F^2/\omega^2$ for $g \rightarrow q \rightarrow g$, in non-ordered limits.

DGLAP attaches $1/\omega$ and colour sum to leg with higher p_t
BFKL attaches them to left-hand leg — **inconsistent**

Sensibleness requirement on matrix formulation.

Use similarity transform S to reattach colour and $1/\omega$ factors in anticollinear limit, so as to restore compatibility between DGLAP and BFKL. Resulting symmetry is

$$\mathcal{K}(1 + \omega - \gamma, \omega) = S(\omega)\mathcal{K}^T(\gamma, \omega)S^{-1}(\omega) .$$

Choose S, for convenience, such that

$$\mathcal{K}^T(\gamma, \omega) = S(\omega)\mathcal{K}^T(\gamma, \omega)S^{-1}(\omega) \implies \mathcal{K}(1 + \omega - \gamma, \omega) = \mathcal{K}(\gamma, \omega)$$

Other requirements

▸ K_{qq}, K_{qg} should be free of $1/\omega$ divergences at all orders
▸ K_{gq}, K_{gg} may at most have $1/\omega$ divergences
▸ No terms in K_{ab} should have any collinear divergence stronger than $1/\gamma$.

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Structure quite similar to single-channel; LLx-LO is:

$$\mathcal{K}_0(\gamma,\omega) = \begin{pmatrix} \Gamma_{qq,0}(\omega)\chi_c^\omega(\gamma) & \Gamma_{qg,0}(\omega)\chi_c^\omega(\gamma) + \Delta_{qg}(\omega)\chi_{\mathrm{ht}}^\omega(\gamma) \\ \Gamma_{gq,0}(\omega)\chi_c^\omega(\gamma) & \Gamma_{gg,0}(\omega)\chi_c^\omega(\gamma) + \dfrac{2C_A}{\omega}\left[\chi_0^\omega(\gamma) - \chi_c^\omega(\gamma)\right] \end{pmatrix}$$

Note $\Delta_{qg}(\omega)$ term: allows one to set *factorisation scheme* at NLO, by modifying the *higher-twist* part of the $\mathcal{K}_{qg}$ kernel.

Without having to add α_s^2/ω term to $\mathcal{K}_{1,qg}$

NB: We choose $\overline{\mathrm{MS}}$

Higher orders:

▶ Add on $\mathcal{K}_1(\gamma,\omega)$ to get NLx-NLO.

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Known limitations

► $\overline{\text{MS}}$ scheme for α_s^n/ω^{n-1} terms in P_{qq}, P_{qg}, P_{gg} only set up to some fixed order (NLO, NNLO), even though known [Catani & Hautmann '94] to all orders.   Believed to be no larger than renorm-scale uncertainties
Based on study of P_{gg}, CCSS '06

► Formalism 'predicts' that at NLx accuracy, at NNLO

$$\Gamma_{gq,2}^{\text{NL}x} = \frac{C_F}{C_A}\Gamma_{gg,2}^{\text{NL}x}$$

But true $\overline{\text{MS}}$ [MVV '04] result differs by an N_c-suppressed term

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Intercept at fixed coupling

Power of growth of cross-sections and splitting functions at fixed coupling.
Rather similar to 2003 results:
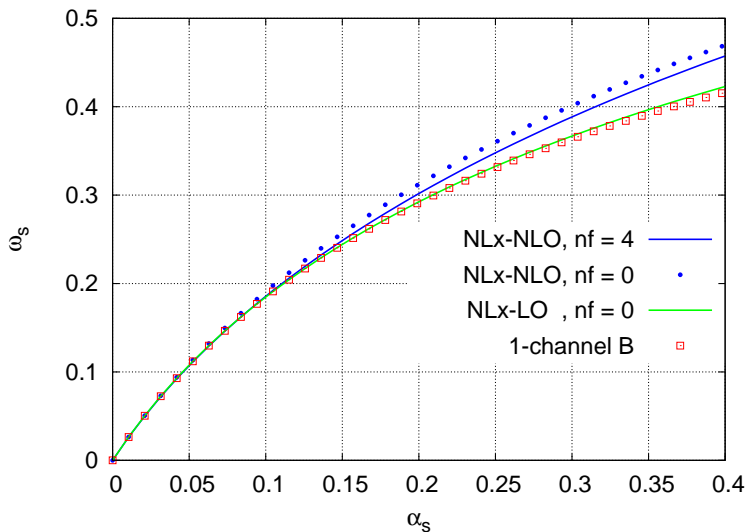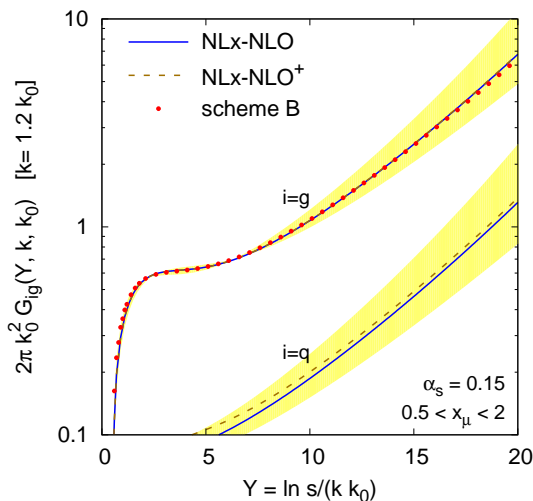
Green function for gluon is very similar to 2003 results.

Scale uncertainties (band) under control

Additionally *generate quark component*, with same power-growth, but suppressed by $\sim \alpha_s$.

Scale uncertainties larger — radiative generation

NNLO part of NLx scheme terms (NLO$^+$) have little impact.

Splitting functions
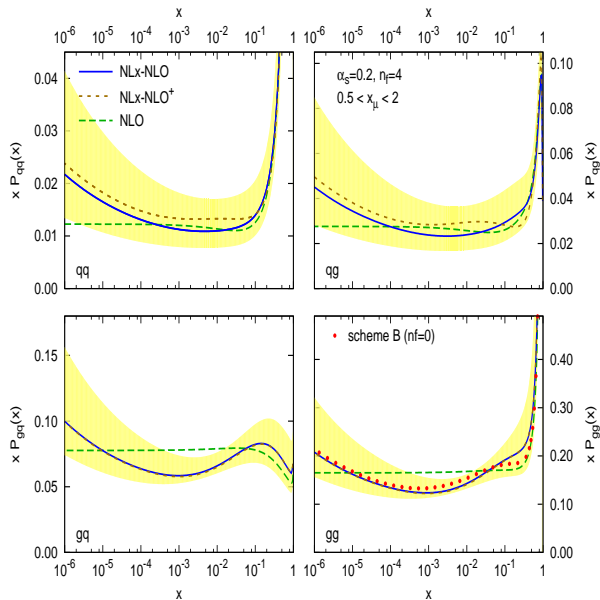
In *gg* channel results again similar to those from 2003

gq channel rather similar to *gg*

> Both have **dip** at $x \sim 10^{-3}$

qq and *qg* channels have barely any dip, and large scale uncertainties — NL*x* is first order of generation of small-*x* quarks.

Conclusions, outlook

▶ Have double integral equation that contains both NLx BFKL and NLO DGLAP in $\overline{\text{MS}}$ scheme.

▶ From it one can deduce Green functions and matrix of effective small-x resummed splitting functions.

▶ Gluon-channel results agree with earlier resummations, now also get full singlet matrix.

Many options available for future

▶ providing splitting functions in convenient form for general use

▶ understanding what happens at NNLO

▶ extending treatment to coefficient functions

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EXTRAS

Building up the kernel. . .

Examine $\bar{\alpha}_s \chi(\gamma)$
    minimum = BFKL power

$$\chi(\gamma) = \underbrace{\chi_0(\gamma)}_{LL} + \underbrace{\bar{\alpha}_s \chi_1(\gamma)}_{NLL} + \dots$$

▶ NLL terms *pathologically large.*
  minimum → max. (unstable)
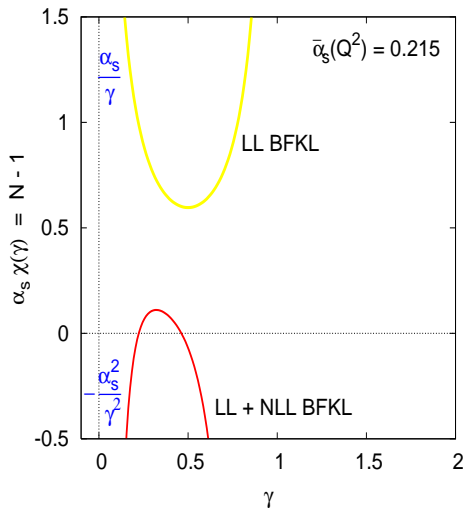   oscillating X-sctns, . . .

▶ Culprit: ⊥ DGLAP logs

  $$\frac{\bar{\alpha}_s}{\gamma} - \frac{11}{12}\frac{\bar{\alpha}_s^2}{\gamma^2} + \dots$$
                $[\gamma^{-1} \Leftrightarrow \ln Q^2]$

▶ Known at *all orders* $(\gamma \to 0)$

  $$\approx \frac{\bar{\alpha}_s}{\bar{\alpha}_s + \gamma}$$ 'Rotated $\gamma(N)$'

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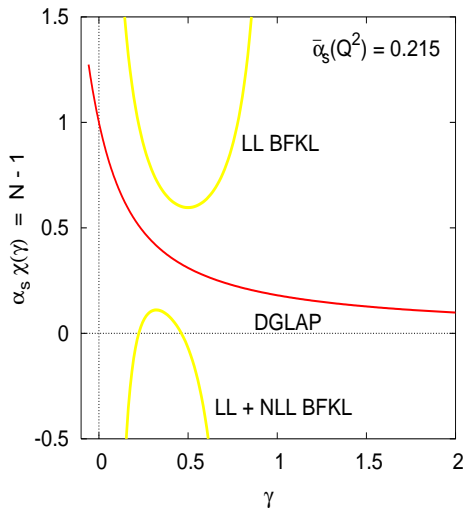minimum → max. (unstable)
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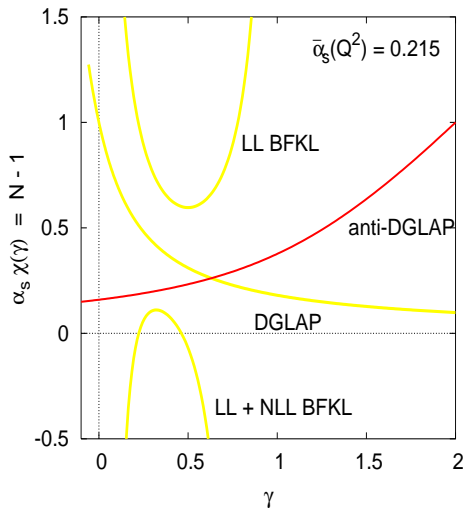   oscillating X-sctns, . . .

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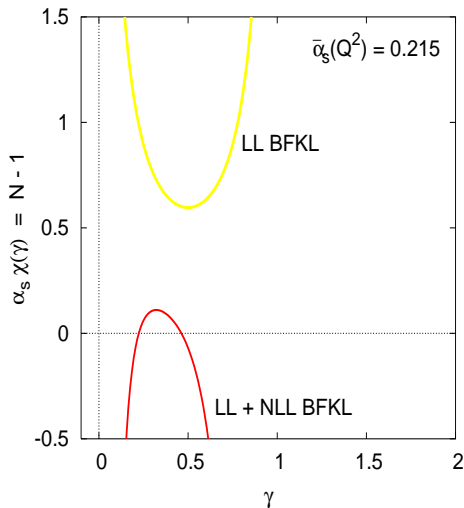
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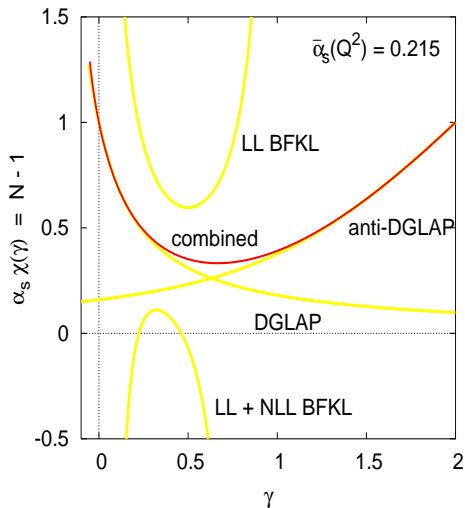 minimum → max. (unstable)
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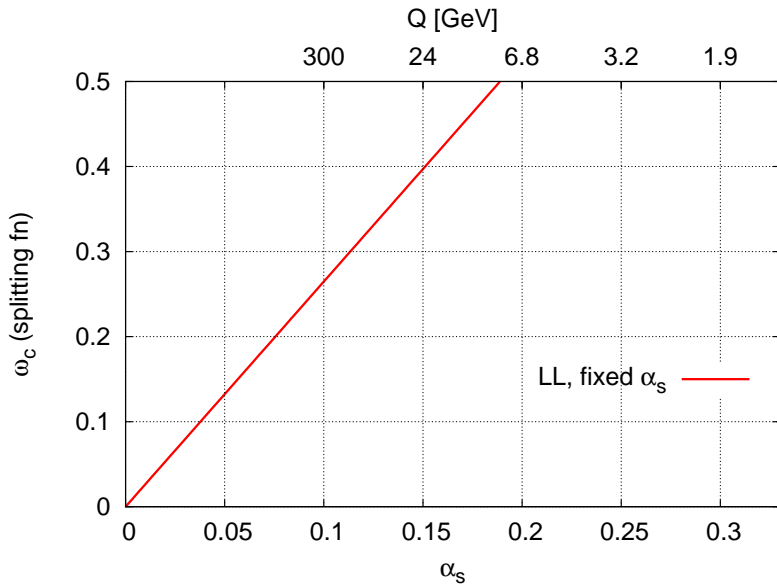minimum = BFKL power

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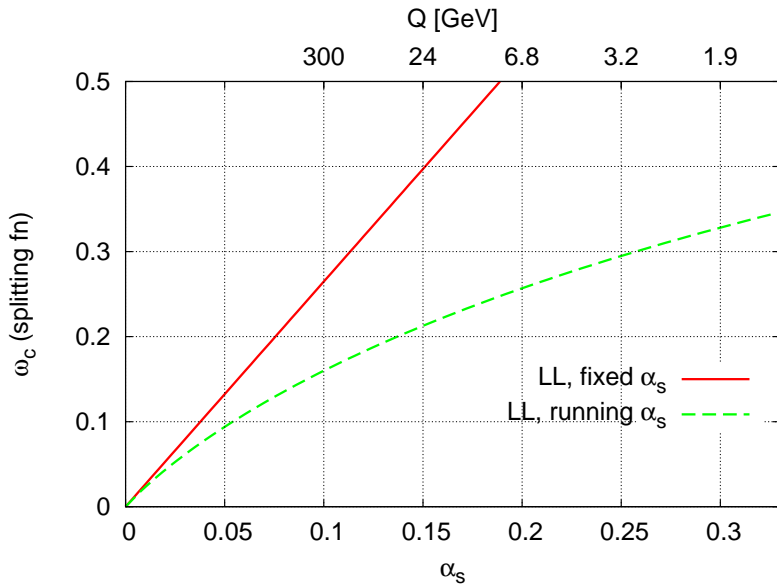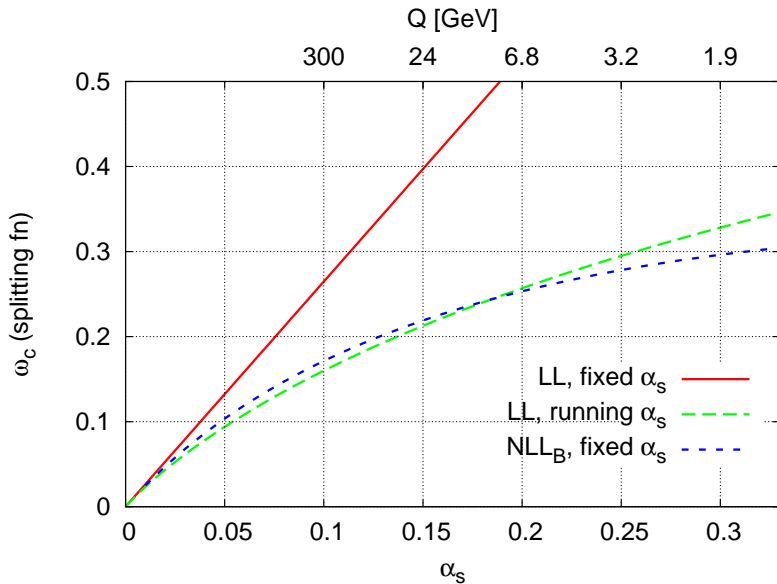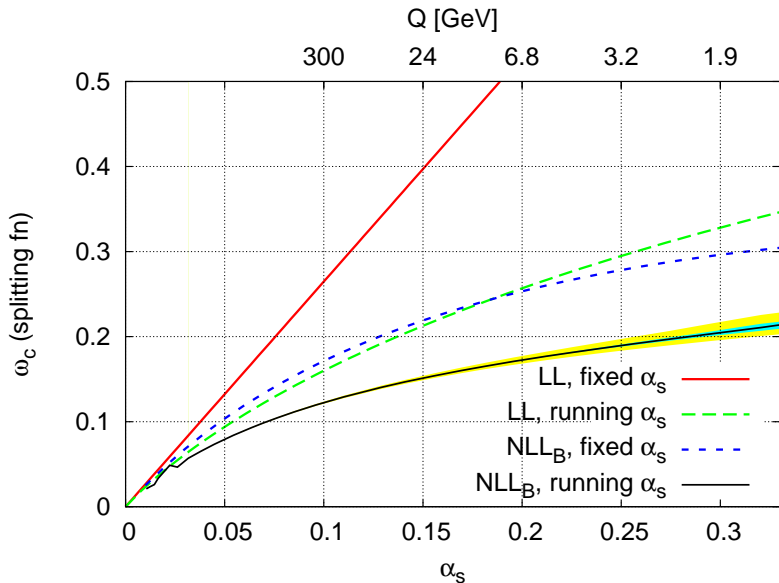
Assemble all constraints:

stable, sensible kernel

Ciafaloni, Colferai, GPS & Staśto;

Altarelli, Ball & Forte; '99–'05

$$S = \begin{pmatrix} 2n_f N_c f_q(\omega) & 0 \\ 0 & (N_c^2 - 1)f_g(\omega) \end{pmatrix},$$

$$\overline{\Gamma} = S\Gamma^T S^{-1} = \begin{pmatrix} \Gamma_{qq} & \frac{n_f}{C_F}\frac{f_q(\omega)}{f_g(\omega)}\Gamma_{gq} \\ \frac{C_F}{n_f}\frac{f_g(\omega)}{f_q(\omega)}\Gamma_{qg} & \Gamma_{gg} \end{pmatrix}.$$

$$\mathcal{K} \simeq \frac{\Gamma}{\gamma} + \frac{\overline{\Gamma}}{1 + \omega - \gamma},$$

$$f_q(\omega) = \frac{2T_R}{\omega + 3} \implies \overline{\Gamma} = \Gamma,$$

$$\mathcal{K}(\alpha_{\mathsf{s}}, \gamma, \omega) \equiv \sum_{n,m,p=0}^{\infty} {}_p\mathcal{K}_n^{(m)} \, \hat{\alpha}^{n+1} \gamma^{m-1} \omega^{p-1} \,, \qquad \hat{\alpha} \equiv \frac{\alpha_{\mathsf{s}}}{2\pi}$$

$$\mathcal{K}_1 = \left(\Gamma_1 - \mathcal{K}_0^{(1)}\mathcal{K}_0^{(0)}\right)\chi_c^\omega + (2C_A)^2\left(\frac{1}{\omega} - \frac{2}{1+\omega}\right)\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\chi}_1^\omega - \tilde{\chi}_1^{(0)}\chi_c^\omega \end{pmatrix}$$

$$\tilde{\chi}_1^{\omega=0} \equiv \tilde{\chi}_1 = \frac{{}_0\mathcal{K}_{gg,1}}{(2C_A)^2} = K_1^{\mathrm{BFKL}} - \frac{\left[{}_0\mathcal{K}_0 \, {}_1\mathcal{K}_0\right]_{gg}}{(2C_A)^2}$$