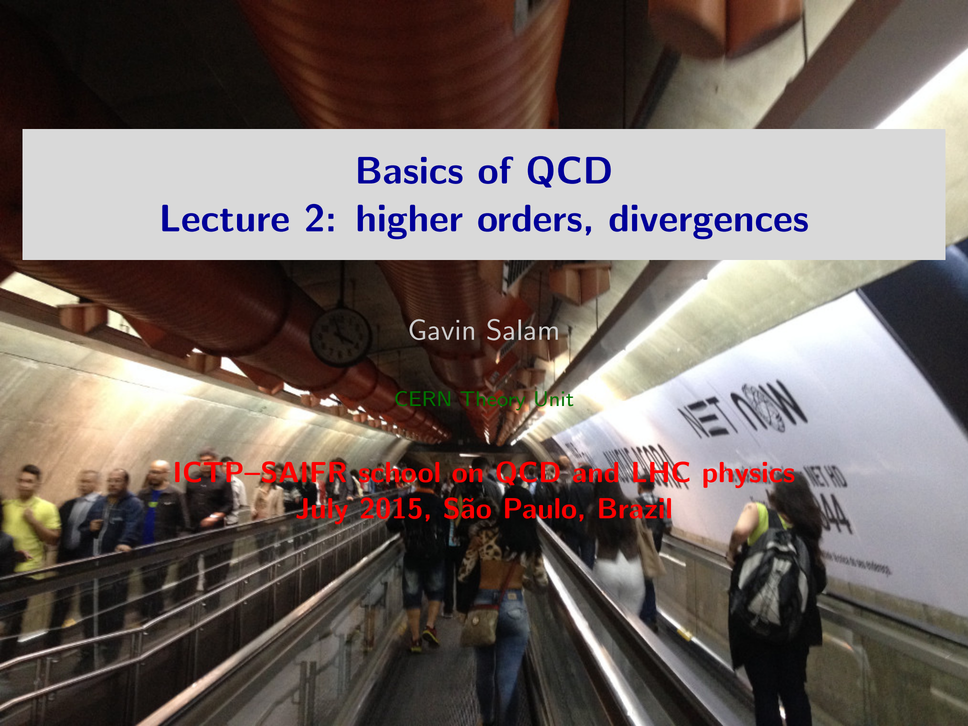

Basics of QCD
Lecture 2: higher orders, divergences

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CERN Theory Unit

ICTP–SAIFR school on QCD and LHC physics
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Neither lattice QCD nor perturbative QCD can offer a full solution to using QCD at colliders

What the community has settled on is

1) **factorisation** of initial state non-perturbative problem

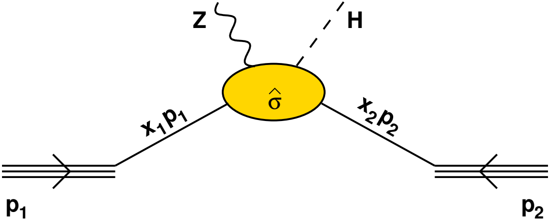
from

2) the "**hard process**," calculated perturbatively

supplemented with

3) non-perturbative modelling of final-state hadronic-scale processes ("**hadronisation**").

Cross section for some hard process in hadron-hadron collisions



$$\sigma = \int dx_1 f_{q/p}(x_1, \mu^2) \int dx_2 f_{\bar{q}/p}(x_2, \mu^2)\, \hat{\sigma}(x_1 p_1, x_2 p_2, \mu^2)\,, \quad \hat{s} = x_1 x_2 s$$

- Total X-section is *factorized* into a 'hard part' $\hat{\sigma}(x_1 p_1, x_2 p_2, \mu^2)$

  Calculated, e.g. with methods discussed in many of the other courses

- and parton distribution functions (PDFs): $f_{q/p}(x, \mu^2)$ is the probability of finding a quark q inside a proton p, and carrying a fraction x of its momentum.

  Determined experimentally, cf. later

  [For now, don't worry about μ^2 "factorisation scale" argument]

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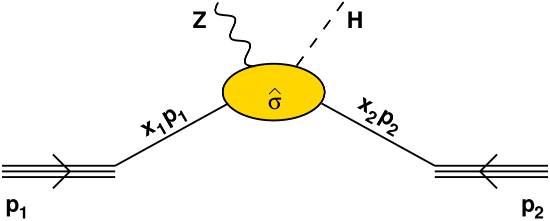

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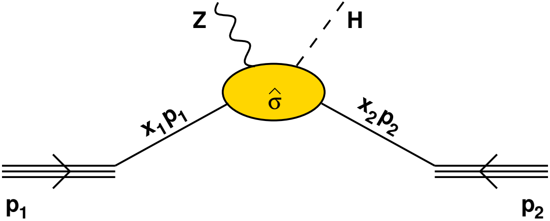

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Factorisation is a term that has several related meanings in QCD.

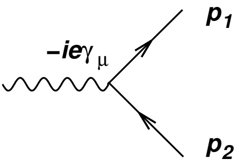
Intimately connected with **infrared divergences**

We can start understanding those by studying a process that's simpler than hadron collisions: e^+e^- collisions with hadronic final states.

Soft gluon amplitude

Start with $\gamma^* \to q\bar{q}$:

$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2)$$



Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$

Make gluon *soft* $\equiv k \ll p_{1,2}$; ignore terms suppressed by powers of k:

$$\mathcal{M}_{q\bar{q}g} \simeq \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k}\right)$$

$\not{p} v(p) = 0,$
$\not{p}\not{k} + \not{k}\not{p} = 2p.k$

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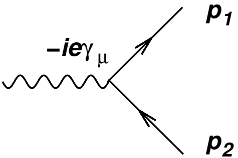

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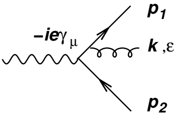

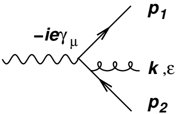

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$$\bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) = -i g_s \bar{u}(p_1) \not{\epsilon} \frac{\not{p}_1 + \not{k}}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\not{A}\not{B} = 2A.B - \not{B}\not{A}$:

$$= -i g_s \bar{u}(p_1)[2\epsilon.(p_1+k) - (\not{p}_1 + \not{k})\not{\epsilon}] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\bar{u}(p_1)\not{p}_1 = 0$ and $k \ll p_1$ (p_1, k massless)

$$\simeq -i g_s \bar{u}(p_1)[2\epsilon.p_1] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

$$= -i g_s \frac{p_1.\epsilon}{p_1.k} \underbrace{\bar{u}(p_1) e_q \gamma_\mu t^A v(p_2)}_{\text{pure QED spinor structure}}$$

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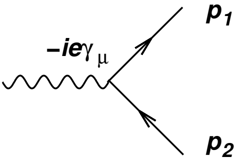

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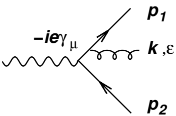

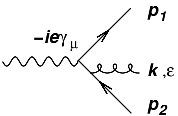

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Squared amplitude

$$|M^2_{q\bar{q}g}| \simeq \sum_{A,\mathrm{pol}} \left| \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

$$= -|M^2_{q\bar{q}}| C_F g_s^2 \left(\frac{p_1}{p_1.k} - \frac{p_2}{p_2.k} \right)^2 = |M^2_{q\bar{q}}| C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

Include phase space:

$$d\Phi_{q\bar{q}g}|M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}}|M^2_{q\bar{q}}|) \underbrace{\frac{d^3\vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}}_{dS}$$

Note property of **factorisation** into hard $q\bar{q}$ piece and soft-gluon emission piece, dS.

$$dS = EdE\, d\cos\theta \frac{d\phi}{2\pi} \cdot \frac{2\alpha_s C_F}{\pi} \frac{2p_1.p_2}{(2p_1.k)(2p_2.k)}$$

$\theta \equiv \theta_{p_1 k}$
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Soft & collinear gluon emission

Take squared matrix element and rewrite in terms of E, θ,

$$\frac{2p_1.p_2}{(2p_1.k)(2p_2.k)} = \frac{1}{E^2(1-\cos^2\theta)}$$

So final expression for soft gluon emission is

$$d\mathcal{S} = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

NB:

- It *diverges* for $E \to 0$ — *infrared (or soft) divergence*
- It *diverges* for $\theta \to 0$ and $\theta \to \pi$ — *collinear divergence*

Soft, collinear divergences derived here in specific context of $e^+e^- \to q\bar{q}$

But they are a very general property of QCD

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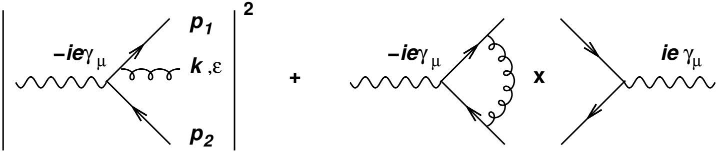
If probability of gluon emission diverges, then how can you calculate anything beyond leading order?

Kinoshita-Lee-Nauenberg theorem tells as that if you sum over allowed states, then result must be finite.

Real-virtual cancellations: total X-sctn

Total cross section: sum of all real and virtual diagrams



Total cross section must be *finite*. If real part has divergent integration, so must virtual part. (Unitarity, conservation of probability)

- $R(E/Q, \theta)$ parametrises real matrix element for hard emissions, $E \sim Q$.
- $V(E/Q, \theta)$ parametrises virtual corrections for all momenta (a "physical fudge" — exact way is to do calc. in dim. reg.)

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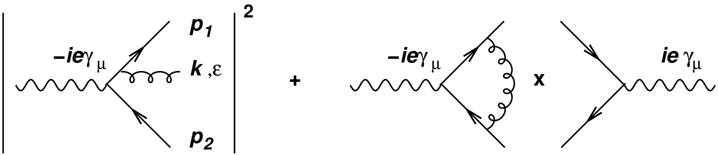

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$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} R(E/Q,\theta)\right.$$

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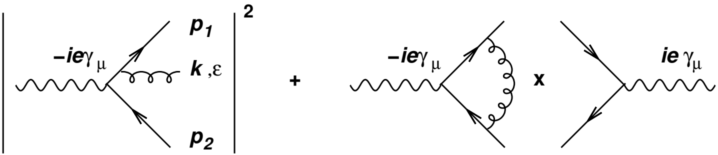

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Total X-section (cont.)

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} \left(R(E/Q,\theta) - V(E/Q,\theta) \right) \right)$$

- From calculation: $\lim_{E \to 0} R(E/Q,\theta) = 1$.
- For every divergence $R(E/Q,\theta)$ and $V(E/Q,\theta)$ should cancel:

$$\lim_{E \to 0} (R - V) = 0\,, \qquad \lim_{\theta \to 0,\pi} (R - V) = 0$$

Result:

- corrections to σ_{tot} come from hard ($E \sim Q$), large-angle gluons
- Soft gluons don't matter:

- Correct renorm. scale for α_s: $\mu \sim Q$ — perturbation theory valid

Total X-section (cont.)

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} \left(R(E/Q,\theta) - V(E/Q,\theta)\right)\right)$$

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Total cross section at NLO

Our treatment so far was a bit rough: designed to emphasize physical nature of divergences.

In practice calculations will be done in $4 + \epsilon$ dimensions and infrared divergences translate to powers of $1/\epsilon$.

Full final answer for σ_{tot} at next-to-leading order (NLO) is, for massless quarks,

$$\sigma_{\text{tot}} = \sigma_{q\bar{q}} \left(1 + \frac{3}{4}\frac{\alpha_s C_F}{\pi} + \mathcal{O}\left(\alpha_s^2\right)\right)$$

Beyond NLO: e.g. NNLO

x	x	x	x	x	x	x	**0 loops (tree–level)**
o	o	o	o				**1 loop**
ø	ø						**2 loops**
0	**1**	**2**	**3**	**4**	**5**	**6**	

ij → Z + n partons

To get N^{p}LO you need the Born (LO) diagram supplemented with all combination of n loops and $p - n$ extra emissions, with $0 \le n \le p$.

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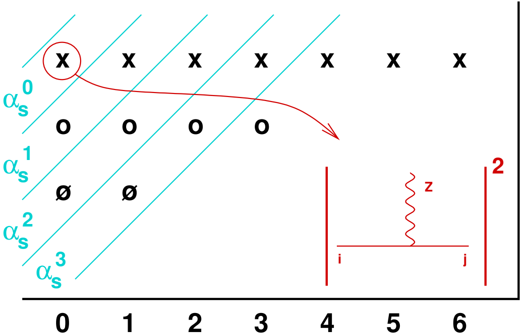

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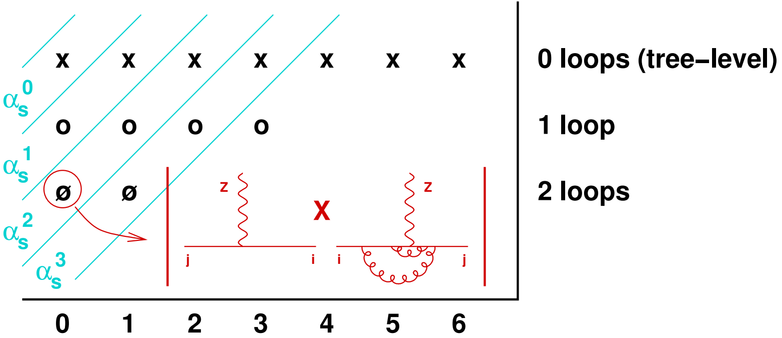

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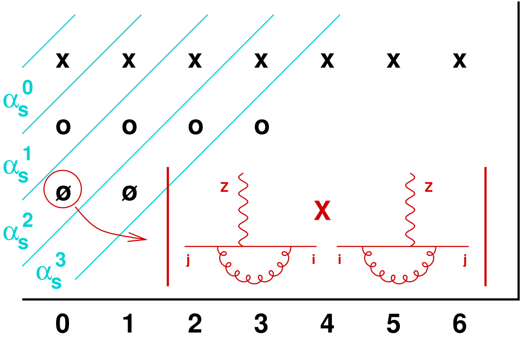

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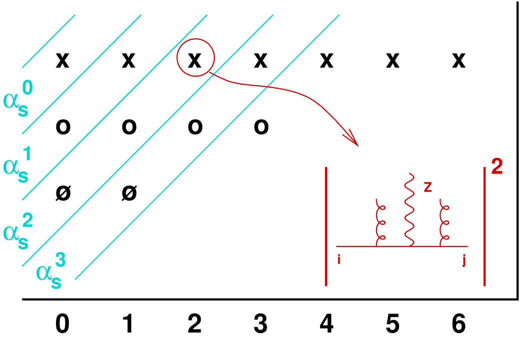

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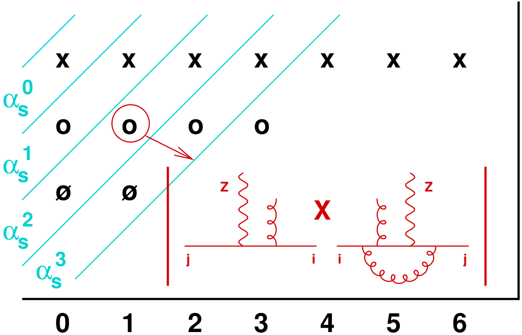

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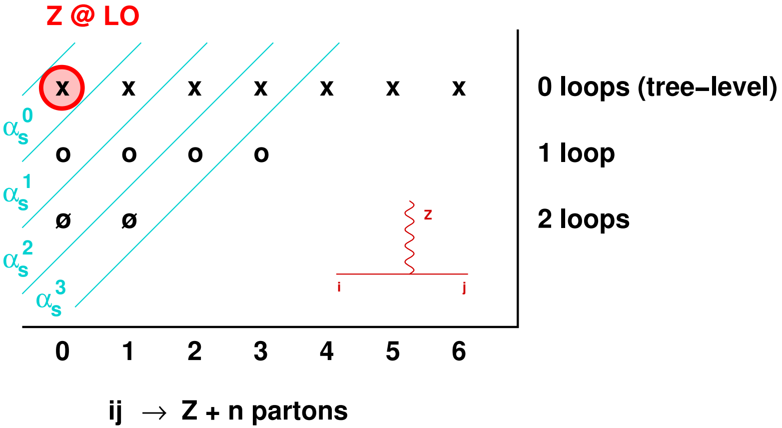

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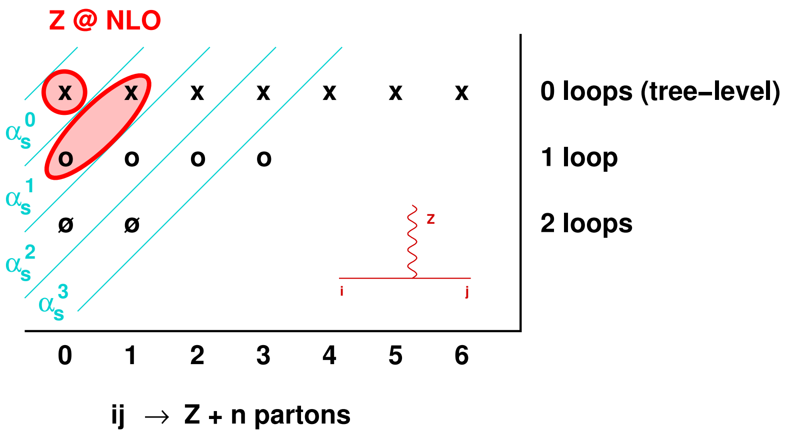

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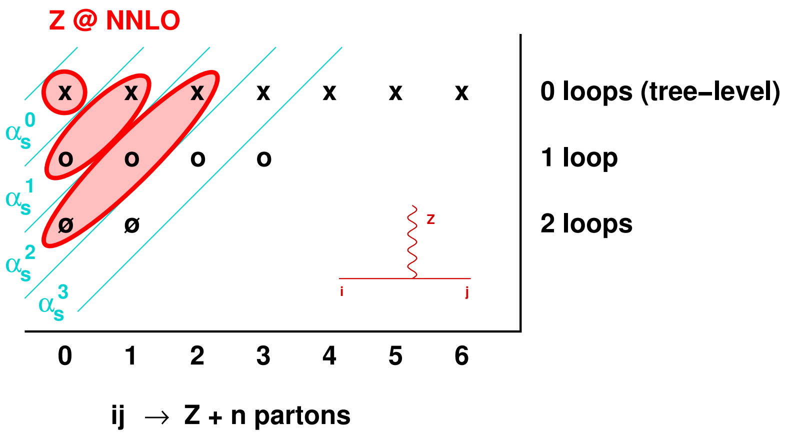

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total X-section (cont.)

Dependence of total cross section on only *hard* gluons is reflected in 'good behaviour' of perturbation series:

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + 1.045 \frac{\alpha_s(Q)}{\pi} + 0.94 \left(\frac{\alpha_s(Q)}{\pi}\right)^2 - 15 \left(\frac{\alpha_s(Q)}{\pi}\right)^3 + \right.$$
$$\left. + \mathcal{O}\left(\alpha_s^4\right) + \mathcal{O}\left(\frac{\Lambda^4}{Q^4}\right)\right)$$

(Coefficients given for $Q = M_Z$)

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Exercise: substitute $\alpha_s(M_Z) = 0.118$ to get a feel for the quality of the expansion.

Question: did we have to write the result as a function of $\alpha_s(Q)$?
Actually, it is standard to write results as a function of $\alpha_s(\mu_R)$, where μ_R is the **renormalisation scale**, to be taken $\mu_R \sim Q$.

Scale dependence

Let's express NLO results for arbitrary μ_R in terms of $\alpha_s(Q)$:

$$\sigma^{\text{NLO}}(\mu_R) = \sigma_{q\bar{q}}\,(1 + c_1\,\alpha_s(\mu_R))$$
$$= \sigma_{q\bar{q}}\left(1 + c_1\,\alpha_s(Q) - 2c_1 b_0 \ln\frac{\mu_R}{Q}\,\alpha_s^2(Q) + \mathcal{O}\left(\alpha_s^3\right)\right)$$

As we vary the renormalisation scale μ_R, we introduce $\mathcal{O}\left(\alpha_s^2\right)$ pieces into the X-section. I.e. generate some set of NNLO terms $\sim$ uncertainty on X-section from missing NNLO calculation.

If we now calculate the full NNLO correction, then it will be structured so as to cancel the $\mathcal{O}\left(\alpha_s^2\right)$ scale variation

$$\sigma^{\text{NNLO}}(\mu_R) = \sigma_{q\bar{q}}\left[1 + c_1\,\alpha_s(\mu_R) + c_2(\mu_R)\alpha_s^2(\mu_R)\right]$$

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Remaining uncertainty is now $\mathcal{O}\left(\alpha_s^3\right)$.

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$$= \alpha_s(Q) - 2b_0\,\alpha_s^2(Q)\,\ln \mu_R/Q + \mathcal{O}\left(\alpha_s^3\right)$$

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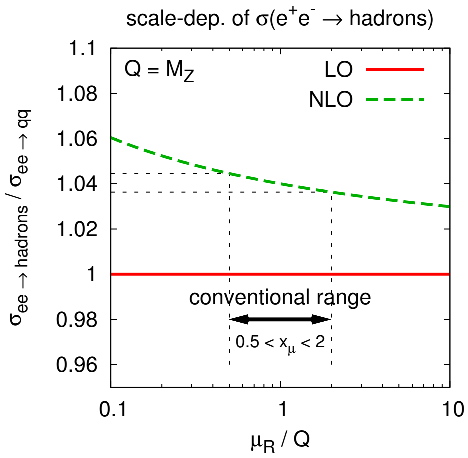

See how at NNLO, scale dependence is much flatter, final uncertainty much smaller.

Because now we neglect only α_s^3 instead of α_s^2

Moral: not knowing exactly how to set scale → blessing in disguise, since it gives us handle on uncertainty.

Scale variation ≡ standard procedure
Beyond LO, often a good guide
But not foolproof!

NB: if we had a large number of orders of perturbation theory, scale dependence would just disappear.

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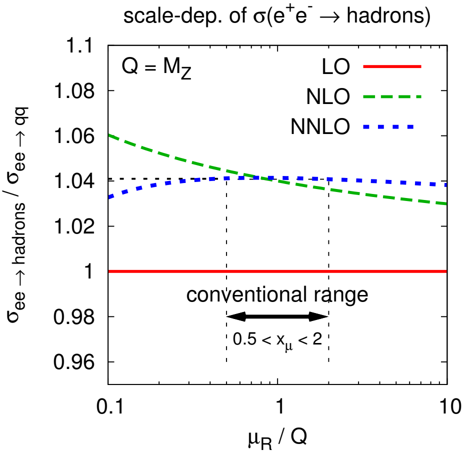

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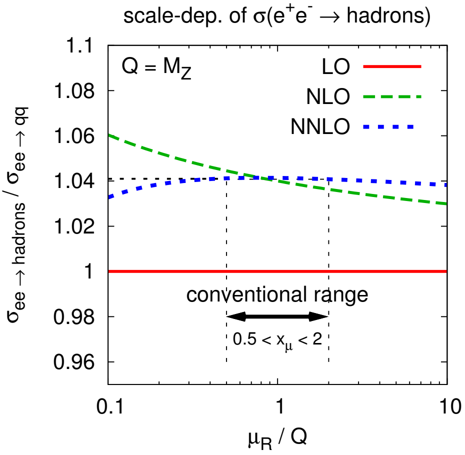

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Exercise: scale dependence

Suppose you have a geometric perturbative series,

$$\sigma = \sigma_0 \sum_{i=0}^{\infty} c^i \alpha_s^i$$

Working in a limit where $\alpha_s \ll 1$, $c \gg 1$ and $c\alpha_s < 1$, evaluate the scale dependence on the estimate for σ obtained when the series is truncated at order n.

Is that scale dependence a good indication of the size of missing higher order terms?

Where to now?

There are two directions we can explore

1. what happens with a more complicated initial state
2. what happens when we look in more detail at the final state