Taking QCD beyond fixed order perturbation theory – systematically

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Today’s colliders are QCD machines

Current and forthcoming high-energy colliders:

<table>
<thead>
<tr>
<th>HERA</th>
<th>Tevatron</th>
<th>LHC</th>
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<tbody>
<tr>
<td>$e^\pm p$</td>
<td>$\bar{p}p$</td>
<td>$pp$</td>
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All involve protons — understanding what’s going on unavoidably involves QCD

Tevatron: main ‘new’ object of study is top quark, interest is in checking its couplings and measuring its mass (e.g. implications for Higgs).

LHC: don’t yet know what ‘new’ objects will be — but ability to extract them from (QCD) backgrounds and measure their properties will almost certainly be limited by the quality of our understanding of QCD.

So where’s the problem? It’s just Feynman diagrams…
Real events bear superficial resemblance to perturbative picture

But

(a) **Fundamental problem**: want a better understanding of correspondence between (i) the perturbative language used for calculations and (ii) the hadrons that are observed.

(b) To get the most out of QCD events for doing ‘other physics’ (searches etc.) → understand, quantitatively, how they differ from naive Feynman diags.

E.g. how do you relate the true mass of a new particle to the mass measured by isolating the jets it decays into?
One way of improving situation is by

Refining our understanding of perturbative QCD

- Next-to-Next-to-Leading-Order (NNLO), multi-leg NLO  Much activity
- Approximations to the behaviour of QCD at all orders  This talk

When discussing new techniques, it’s useful to have a playground:

- Simple collider environments: $e^+e^-$ (LEP), DIS (HERA).
- Special observables: event shapes — measures of deviation from idealised lowest order Feynman diagrams.
- Then apply understanding to real analyses at hadron colliders

This talk will examine principles of all-order calculations in the simplest possible environment ($e^+e^- \rightarrow 2\text{jets}$), attempting to illustrate lessons that hold in general.
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Perturbative QCD at fixed orders
- Soft and collinear divergences
- Infrared and collinear safety $\leftrightarrow$ (pseudo)-convergent perturbation series

Fixed-order breakdown, all-order log-enhanced structures
- fixed orders insufficient for describing most common events
- understanding of divergences $\leftrightarrow$ all-order rearrangement of perturbation series

Resummation done systematically
- issues
  - *recursive* infrared collinear safety
  - automated resummation
Consider Feynman diagram (c.o.m. energy = $Q$)

$$p_2 \xrightarrow{k} p_1$$

Simplest limit:
- emitted gluon has small energy $E_k \ll Q$ (soft)
- is at small angle wrt quark, $\theta \ll 1$ (collinear)

Propagator goes on-shell $\leftrightarrow$ divergence:

$$d\Phi_{q\bar{q}g}|M_{q\bar{q}g}^2| \simeq d\Phi_{q\bar{q}}|M_{q\bar{q}}^2| \cdot \frac{8}{3} \frac{\alpha_s}{\pi} \cdot \frac{dE_k}{E_k} \frac{d\theta}{\theta}$$

Such soft and collinear divergences are pivotal in this talk.
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\times \\
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Such \textit{soft and collinear divergences} are pivotal in this talk.
Probability of emitting 1 gluon

Based on soft-collinear limit, probability for emitting 1 gluon is

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\text{Prob}(1 \text{ gluon}) \sim \frac{16}{3} \frac{\alpha_s}{\pi} \int_0^Q \frac{dE}{E} \int_0^{\pi/2} \frac{d\theta}{\theta}
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This is \textit{infinite}. Perhaps integrals should not go below non-perturbative scale \( \Lambda \)?

Put cut-off:

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\text{Prob}(1 \text{ gluon}) \sim \frac{16}{3} \frac{\alpha_s}{\pi} \int_\Lambda^Q \frac{dE}{E} \int_\Lambda/Q^{\pi/2} \frac{d\theta}{\theta} \sim \frac{16}{3\pi} \alpha_s \ln^2 \frac{Q}{\Lambda}
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Two large logarithms, one ‘soft’, one ‘collinear’ (both depend on cutoff).

Does small coupling save us? \( \alpha_s = 1/(b_0 \ln Q/\Lambda) \):

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Instead of calculating ‘flow of gluons’, let’s try and look at flow of energy.

E.g. ‘jet broadening’, $B_T$ (transverse momentum flow wrt jet axis)

\[ B_T = \frac{1}{2Q} \sum_i |\vec{q}_i \times \vec{n}| \approx \frac{E_k \theta}{Q} \quad (\theta \ll 1) \]

Do perturbative calculation for mean value of broadening:

\[ \langle B_T \rangle \sim \frac{16}{3} \frac{\alpha_s}{\pi} \int_0^Q \frac{dE}{E} \int_0^{\pi/2} \frac{d\theta}{\theta} \cdot \frac{E\theta}{Q} \]

Divergences are cancelled, because ‘observable’ ($B_T$) vanishes when the gluon is soft or collinear. Result is truly perturbative.

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Suitable observables are straightforwardly calculable.
Crucial property of broadening was that effect of an additional gluon vanished \( \propto \) a power of its softness and collinearity.

Infrared and collinear (IRC) safety

For an observable’s distribution to be calculable in perturbation theory, the observable should be infra-red [and collinear] safe, i.e. insensitive to the emission of soft or collinear gluons. In particular if \( \vec{p}_i \) is any momentum occurring in its definition, it must be invariant under the branching

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\vec{p}_i \rightarrow \vec{p}_j + \vec{p}_k
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whenever \( \vec{p}_j \) and \( \vec{p}_k \) are parallel [collinear] or one of them is small [infrared].

[QCD and Collider Physics (Ellis, Stirling & Webber)]
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Sterman & Weinberg ’77

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Next: calculate higher-order corrections. At each order, probability of emitting gluon $\gg 1 \rightarrow$ complex configurations with many gluons:

But: high multiplicity comes from soft, collinear region – these gluons don’t affect observable (IRC safety), and cancel nearly fully with virtual corrections.

Field theory: real-virtual cancellation
Observable: IRC safety

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**Field theory:** real-virtual cancellation

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Combination of field theory and observable properties allow us to *pretend* that the real world looks like perturbation theory.
Consider pure $\alpha_s^2$ contributions. Conceptually simple:

In practice
- Physicist calculates matrix elements once $\rightarrow$ into computer program.
- Program generates random configurations (real & virtual), calculates arbitrary IRC-safe observable (subroutine), weights with matrix elements.

Subtlety: how do you combine
- observable in 4-dimensions,
- matrix elements in $4 + 2\epsilon$ dimensions (dim.-reg.)?

General NLO solution: Catani & Seymour ’96 + Dittmaier & Trocsanyi ’02
First NNLO solution: Gehrmann-De Ridder, Gehrmann & Glover ’05
QCD beyond fixed order (p. 11)

Next-to-leading order (NLO) predictions

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Broadening distribution at NLO

QCD beyond fixed order (p. 12)
— NLO breakdown, all-order log structure

Total Broadening ($B_T$)

$\alpha_s + \alpha_s^2$

Large $B_T$
✓ Shape OK
✓ Normalisation correct

Small $B_T$
✗ Shape wrong (divergent)
✗ This is where you have most data

OPAL 91 GeV
QCD beyond fixed order (p. 12)

NLO breakdown, all-order log structure

Broadening distribution at NLO

![Graph showing OPAL 91 GeV data with LO predictions.]

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$\alpha_s + \alpha_s^2$

Total Broadening ($B_T$)
What’s happening?

What is probability, $\Sigma(B)$, that broadening $< \text{some value } B$?

$$\Sigma(B) \sim 1 + \frac{16}{3} \frac{\alpha_s}{\pi} \int_0^\infty dE \frac{d\theta}{\theta} \Theta(B - \frac{E\theta}{Q}) - \frac{16}{3} \frac{\alpha_s}{\pi} \int_0^\infty dE \frac{d\theta}{\theta}$$

$$\sim 1 - \frac{16}{3} \frac{\alpha_s}{\pi} \int_0^\infty dE \frac{d\theta}{\theta} \Theta(\frac{E\theta}{Q} - B) \sim 1 - \frac{8}{3} \frac{\alpha_s}{\pi} \ln^2 B$$

Double logarithm due to incomplete real-virtual cancellation of soft and collinear divergences, when considering narrow jets.

NB: resulting distribution diverges

$$\frac{d\Sigma}{dB} \sim \frac{16}{3} \frac{\alpha_s}{\pi} \ln \frac{1}{B} \frac{1}{B}$$
Examine soft-collinear limit of two gluons:

Two propagators nearly on-shell $\leftrightarrow$ 4 divergences ($E_a \ll E_b$). Can be viewed as two parts (approx.):

- independent emission of two gluons (diags, 1,3)
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All diagrams could potentially give us $\alpha_s^2 \ln^4 B$
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What will happen at next order?

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Normal perturbative expansion is fine in formal perturbative $\alpha_s \to 0$ limit

$$\Sigma(B) = 1 + \alpha_s f_1(B) + \alpha_s^2 f_2(B) + \ldots$$

$$f_n(B) \sim \ln^{2n} B \text{ for } B \ll 1$$

In region where you have most of the data $\ln B \gg 1$ and $\alpha_s^n f_n(B) \sim 1$ — series does not converge.

But origin of logs is simple: residual non-cancellation of real and virtual soft-collinear divergences. Can imagine calculating them at all orders:

$$\Sigma(B) \sim \sum_{n=0}^{\infty} H_{n,2n} \alpha_s^n \ln^{2n} B + \mathcal{O}(\alpha_s^n \ln^{2n-1} B)$$

$$= h_1(\alpha_s L^2) + \sqrt{\alpha_s} h_2(\alpha_s L^2) + \ldots,$$

$L \equiv \ln \frac{1}{B}$

This is a resummation of leading logarithms (LL), $h_1(\alpha_s L^2)$

Will converge even for large values of the logarithm, $\alpha_s L^2 \sim 1$ since $h_1 \sim 1$, $h_2 \sim 1$ [NB: traded $L^{-1}$ for $\sqrt{\alpha_s}$ in front of $h_2$]
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$$\Sigma(B) \sim \sum_{n=0}^{\infty} \frac{H_{n,2n}}{n!} \alpha_s^n \ln^{2n} B + O \left( \alpha_s^n \ln^{2n-1} B \right)$$

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Step 1. Simplify matrix element.

- B measures transverse momentum flow relative to main event ($\sim q\bar{q}$) axis.
- Secondary gluon splitting does not change observable (will cancel fully against virtuals)
- Take only independent emission:

$$d\Phi_n|M^2(k_1, \ldots k_n)| \rightarrow \frac{1}{n!} \prod_{i=1}^{n} \frac{16}{3} \frac{\alpha_s}{\pi} \frac{dE_i}{E_i} \frac{d\theta_i}{\theta_i}$$

Minus corresponding virtual (loop) terms
Step 2. Simplify observable

- Calculate observable with *arbitrary number of emissions*. In soft and collinear limit it ‘simplifies’ to

\[
B = \frac{1}{2Q} \left( \sum_{i=1}^{n} |\vec{k}_{ti}| + \sum_{i \in \text{right}} |\vec{k}_{ti}| + \sum_{i \in \text{left}} |\vec{k}_{ti}| \right)
\]

- For now approximate this as

\[
B = \frac{1}{Q} \max \{ k_{t1}, k_{t2}, \ldots k_{tn} \}
\]

Since \( \ln^2 [B \times \mathcal{O}(1)] = \ln^2 B + \mathcal{O}(1) \cdot \ln B \), this does not change LL.

- Translate to limit on all \( k_{ti} = E_i \theta_i \):

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\Sigma(B) \approx \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \frac{16 \alpha_s}{3} \frac{1}{\pi} \int \frac{dE_i}{E_i} \frac{d\theta_i}{\theta_i} \left[ \Theta(B - E_i \theta_i) - \frac{1}{\text{virt}} \right]
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Exponentiated double logarithms
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Since \(\ln^2[B \times O(1)] = \ln^2 B + O(1) \cdot \ln B\), this does not change LL.

- Translate to limit on all \(k_{ti} = E_i \theta_i\):

\[
\Sigma(B) \simeq \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \frac{16 \alpha_s}{3 \pi} \int \frac{dE_i}{E_i} \frac{d\theta_i}{\theta_i} \left[ \Theta(B - E_i \theta_i) - \frac{1}{\text{virt}} \right] \]

\[
\simeq \exp \left[ -\frac{8 \alpha_s L^2}{3 \pi} \right]
\]

Exponentiated double logarithms
\[ \exp[-\alpha_s L^2] \] is typical of *Sudakov suppression* — if you want broadening to be small, pay the price of *suppressing emission* (i.e. virtual terms).

Exponentiated form does not always hold, e.g. ‘Jade jet resolution,’ \( y_{3J} \):

\[
\Sigma(y_{3J}) = 1 - \frac{4}{3} \frac{\alpha_s L^2}{\pi} + \frac{5}{12} \left( \frac{4}{3} \frac{\alpha_s L^2}{\pi} \right)^2 + \ldots
\]

Brown & Stirling '90

When it *does* hold, \( \exists \) more powerful reorganisation of logs

\[
\Sigma(B) = \exp \left[ \sum_{n=1}^{\infty} G_{n,n+1} \alpha_s^n L^{n+1} + O(\alpha_s^n L^n) \right]
\]

\[
= \exp \left[ L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots \right]
\]

*Better than previous hierarchy*: valid up to \( L \sim 1/\alpha_s \) (rather than \( L \sim 1/\sqrt{\alpha_s} \)) and successive terms suppressed by \( \alpha_s \) (instead of \( \sqrt{\alpha_s} \)).
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  \text{LL} \quad \text{NLL} \quad \text{NNLL}
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Next-to-leading-logarithmic (NLL) accuracy is currently *state of the art* for QCD final-state resummations.

**Ingredients** (in addition to those shown so far):
- Full treatment of observable
- Proper coupling (scheme, two-loop running)
- Careful evaluation of sum over emissions

  - **Pioneered:** Catani, Trentadue, Turnock, Webber (CTTW) ’92
  - **Broadening:** CTW ’92; Dokshitzer, Lucenti, Marchesini & GPS ’98
  - **NB:** simple observable (EEC) recently done at NNLL: de Florian & Grazzini ’04
NLO breakdown, all-order log structure

Broadening distribution at NLO+NLL

NLL shape OKish!

NB: peak is at $\alpha_s L \sim 1$.

Remaining difference ascribed to parton-hadron transition, hadronisation

Only with resummation can hadronisation be separated from perturbative part

\[ \alpha_s + \alpha_s^2 + e^{\alpha_s L^{n+1} B} + \alpha_s^2 L^n B + \frac{1}{Q} \]
QCD beyond fixed order (p. 20)
NLO breakdown, all-order log structure

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Unlike NLO (matrix-element done once, rest done my Monte Carlo), NLL ‘event-shape’ resummation nearly always been done *manually, analytically.*

<table>
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Antonelli, Appleby, Banfi, Berger, Burby, Catani, Dasgupta, Dissertori, Dokshitzer, Glover, Kucs, Lucenti, Marchesini, Oderda, Salam, Schmelling, Seymour, Smye, Sterman, Trentadue, Turnock, Webber, Zanderighi.  
[Since 1992]

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Few practitioners unscathed
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Sources of difficulty (globalness)

**Global observable:**

*e.g.* total $e^+e^-$ Broadening, $B_T$

making $B \ll 1$ restricts emissions everywhere.

Coherence + globalness:

⇒ emissions can be resummed as if independent \((\text{proved})\)

Answers guaranteed to NLL accuracy

**Non-Global observable:**

Right-hemisphere Broadening, $B_R$

making $B_R \ll 1$ restricts emissions in right-hand hemisphere \((H_R)\).

Tempting to assume one can:

- ignore left hemisphere \((H_L)\)
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WRONG AT NLL ACCURACY

Dasgupta & GPS '01
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Dasgupta & GPS '01
QCD beyond fixed order (p. 23)

Resummation done systematically

Issues

Resummation of NG observables

All-orders:

Unrestricted semi-soft gluons (left) change pattern of radiation of large-angle soft gluons (right)

Difficulties, features:

- Logarithms resummed so far only in large-$N_c$ limit
  
  Dasgupta & GPS '01, '02
  Banfi, Marchesini & Smye '02

- In general, boundary between the two regions may have arbitrary shape.

- It may depend on the pattern of emissions (e.g. with jet algo).
  
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- Unexpected relations with BK, BFKL and JIMWLK equations in small-$x$ (high-energy) limit of QCD
  
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Other difficulty is in **handling the soft-collinear limit of the observable:**

- calculate how limit on observable constrains momenta of \( n \) particles
- then express constraint in factorised form, if it exists

E.g.,

\[
\Theta(y_{3C} Q^2 - \max(k_{t1}^2, k_{t2}^2, \ldots, k_{tn}^2)) \rightarrow \prod_{i=1}^{n} \Theta(y_{3C} Q^2 - k_{ti}^2)
\]

\( y_{3C} = 3 \)-jet resolution, Cambridge algorithm

Most cases are more complex

\[
\Theta(\tau Q - k_{t1} - k_{t2} - \ldots - k_{tn}) \rightarrow \int \frac{d\nu}{2\pi i\nu} e^{\nu \tau Q} \prod_{i=1}^{n} e^{-\nu k_{ti}}
\]

\( \tau = \) any thrust-like observable

Some may even be insoluble analytically

\[
\Theta(T_M Q - \max(\vec{k}_{t1} + \vec{k}_{t2} + \ldots + \vec{k}_{tn})) \rightarrow ???
\]

\( T_M = \) thrust-major, done numerically Banfi, GPS & Zanderighi ’01
What we would like:
Something as good as manual analytical resummation

- Guaranteed (verifiable) accuracy, exponentiation
- Separate LL, NLL functions, $g_1(\alpha_s L), g_2(\alpha_s L)$
- Expansions of $g_1$ and $g_2$ to fixed order in $\alpha_s$

Monte Carlo resummation:
Event generators (Herwig, Pythia, …) generate multiple divergent soft-collinear radiation = powerful automated resummation programs!

✓✓ Observable treated exactly $\Leftrightarrow$ very flexible.
✓ Includes hadronisation model
✗ Accuracy sometimes unclear (depends on observable, no NLL for multi-jet processes)
✗ Difficult to estimate uncertainties of calculation
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Generic resummation?

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Follow model of fixed order calculations

Identify combination of

- properties of QCD matrix elements
- requirements on observable

such that a systematic approximation procedure emerges.

NB: will consider only *global* observables, so as to simplify problem.
Use ‘Lund’ representation of kinematic plane: $\ln k_t$ and $\eta = -\ln \tan \theta / 2$
Phase space \((e^+e^- \rightarrow 2 \text{jets})\)

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Introduce observable (\& 1 emission).

Take \textit{general observable}, \( V(p_1, \ldots) \).

Require that it vanish smoothly in soft, collinear limits:

\[ V(p_1, p_2, k) \sim (k_t/Q)^a e^{-b|\eta|} \]

Requirement \( V(\ldots) < v \rightarrow \) boundary of a \textit{vetoed region} for 1 emission

\[ \ln v = a \ln \frac{k_t}{Q} - b|\eta| \]

Diagram shows \( a = b = 1 \)

Real—virtual cancels \textit{everywhere but vetoed region}, leaving:

\[ \Sigma(V < v) = 1 + \underbrace{G_{12} \alpha_s L^2}_{\text{Vetoed area}} + \underbrace{G_{11} \alpha_s L}_{\text{edges}} \]

NB: \( -\alpha_s \frac{dE}{E} \frac{d\theta}{\theta} \sim -\alpha_s d\ln k_t d\eta \)
QCD beyond fixed order (p. 28)

Resummation done systematically

Strategy

Introduce observable (& 1 emission)

vetoed region

hard + collinear: \( \eta = \ln Q/k_t \)

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Diagram shows $a = b = 1$

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Vetoed area edges

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Fixed order: series in $\alpha_s$, so consider limit $\alpha_s \rightarrow 0$ for fixed $V(p_1, \ldots)$

Resummation expansion:

$$\ln \Sigma = \alpha_s^{-1} g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots,$$

so take $\alpha_s \rightarrow 0$ with $\alpha_s L$ constant

For 1 emission, rescaling of $L$ and $\alpha_s$ equivalent to remapping of phase-space:

Question: how does observable behave under such a scaling of momenta when there are many emissions?
QCD beyond fixed order (p. 29)

Resummation done systematically

Strategy

Scaling limit

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Multiple emission properties

- Parametrise emission momenta by effect on observable:

  \[
  \kappa(\bar{v}) \text{ is a momentum such that } V(\{p\}, \kappa(\bar{v})) = \bar{v}
  \]

  A specific function \( \kappa(\bar{v}) \) maps out a path in \( \eta, \ln k_t \) space

- Require observable to scale universally for any number of emissions:

  \[
  \lim_{\bar{v} \to 0} \frac{1}{\bar{v}} V(\{p\}, \kappa_1(\zeta_1 \bar{v}), \kappa_2(\zeta_2 \bar{v}), \ldots) = f(\zeta_1, \zeta_2, \ldots)
  \]

  For any \( \{\zeta_i\} \), and any set of paths \( \{\kappa_i\} \)

This allows us to give meaning to the limit \( \alpha_s \to 0 \) with \( \alpha_s L \) fixed, for any number of emissions — because scaling properties of observable are independent of number of emissions.

All subsequent discussion is to be imagined in this scaling limit.
Multiple emission properties

- Parametrise emission momenta by effect on observable:

  \[ \kappa(\bar{v}) \text{ is a momentum such that } V(\{p\}, \kappa(\bar{v})) = \bar{v} \]

  A specific function \( \kappa(\bar{v}) \) maps out a path in \( \eta, \ln k_t \) space

- **Require** observable to *scale universally* for any number of emissions:

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QCD beyond fixed order (p. 31)
- Resummation done systematically
- Strategy

What happens at all orders?

Problem with arbitrary set of emissions is too complex.

Need to simplify it (like we simplified fixed-order PT at beginning).

Keep just subset of emissions.

But, are we allowed to throw away the remaining emissions?

Only if they don't affect observable and cancel with virtuals in M.E.

Observable

Need condition like IRC safety

- ‘softness’ defined in terms of effect on observable
- soft limit must *commute* with scaling limit

recursive IRC safety
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Matrix element

- anything very soft cancels with corresponding virtual correction
- emissions on disparate angular scales behave independently

QCD coherence
Recall scaling property

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Recursive IRC safety:

- Require \textit{recursive} infrared-collinear safety:
  \[
  \lim_{\zeta_n \to 0} f(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) = f(\zeta_1, \zeta_2, \ldots, \zeta_{n-1})
  \]
  Or:
  \[
  \left[ \lim_{\vec{v} \to 0}, \lim_{\zeta_n \to 0} \right] \frac{1}{\vec{v}} V(\{p\}, \kappa_1(\zeta_1 \vec{v}), \kappa_2(\zeta_2 \vec{v}), \ldots, \kappa_n(\zeta_n \vec{v})) = 0
  \]
Recall scaling property

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Normal v. recursive IRC safety

<table>
<thead>
<tr>
<th></th>
<th>normal IRC safety</th>
<th>recursive IRC safety</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>softness defined relative to</strong></td>
<td>hard scale</td>
<td>soft scale of observable</td>
</tr>
<tr>
<td><strong>problem reduces to one of</strong></td>
<td>low <em>number</em> of emissions</td>
<td>low <em>density</em> of emissions in $\eta$</td>
</tr>
<tr>
<td><strong>allowing use of</strong></td>
<td>fixed-order PT</td>
<td>independent emission approximation</td>
</tr>
</tbody>
</table>

NB: independent emission approximation results from coherence $\equiv$ emissions widely separated in angle are independent.

Coherence recently questioned at subleading $N_c$ and high orders ($\alpha_s^4 L^5$) in $pp \rightarrow 2$ jets

[Forshaw, Kyrieleis and Seymour '06]
QCD beyond fixed order (p. 34)
- Resummation done systematically
- Strategy

**Schematic framework**

Sum over real and virtual emissions in blue band and above is sufficient for any resummation accuracy.

- **LL:** consider just exponential of virtuals in vetoed region:
  \[ \alpha_s L^2 \rightarrow e^{\alpha_s^n L^{n+1}} \]

- **NLL:** need to account for edges
  \[ \alpha_s L \rightarrow e^{\alpha_s^n L^n} \]

In blue band: sum over widely separated individual emissions
- low density \( \sim \alpha_s \); coherence
- \( \rightarrow \) treat them as independent

- **NNLL:** account for corners
  \[ \alpha_s \rightarrow e^{\alpha_s^n L^{n-1}} \]

and 1 correlated pair of emissions (+ any # of indep. emissions)
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A1. formulate exact applicability conditions for the approach (its scope)
A2. derive a master formula for a generic observable in terms of simple properties of the observable

Numerical work (to be repeated for each observable)

N1. let an "expert system" investigate the applicability conditions
N2. it also determines the inputs for a master formula
N3. straightforward evaluation of the master formula, including phase space integration etc.

Note: N1 and N2 are core of automation

a) they will require high precision arithmetic to take asymptotic (soft & collinear) limits

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\[ V(\{p\}, k) = d_\ell \left( \frac{k_t}{Q} \right)^{a_\ell} e^{-b_\ell \eta} g_\ell(\phi). \]

Born momenta soft collinear emission

- Determine coefficients \( a_\ell, b_\ell, d_\ell \) and \( g_\ell(\phi) \) for emissions close to each hard Born parton (leg) \( \ell \).
- Require continuous globalness, i.e. uniform dependence on \( k_t \) independently of emission direction (\( a_1 = a_2 = \cdots = a \)).
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We've mostly discussed soft part, \( \exists \) also a collinear part.
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QCD beyond fixed order (p. 36)

- Resummation done systematically
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Given info from previous pages, **final answer is analytical**:

\[
\ln \Sigma(v) = - \sum_{\ell=1}^{n} C_\ell \left[ r_\ell(L) + r'_\ell(L) \left( \ln \bar{d}_\ell - b_\ell \ln \frac{2E_\ell}{Q} \right) \right] \\
+ B_\ell \ T \left( \frac{L}{a + b_\ell} \right) + \sum_{\ell=1}^{n_i} \ln \frac{q_\ell(x_\ell, e^{-\frac{2L}{a+b_\ell} \mu_f^2})}{q_\ell(x_\ell, \mu_f^2)} \\
+ \ln S(T(L/a)) + \ln \mathcal{F}(C_1 r'_1, \ldots, C_n r'_n),
\]

- \( C_\ell = \text{colour factor; } q_\ell = \text{PDF} \)
- \( r_\ell(L) \Rightarrow \alpha_s^n L^{n+1}; r'_\ell(L), T(L) \Rightarrow \alpha_s^n L^n \)

**Non-trivial parts:**
- \( S(T(L/a)) = \text{large-angle logarithms (proc. dep.)} \)
  - Botts-Kidonakis-Oderda-Sterman '89–'98; Bonciani et al '03
- \( \mathcal{F}(\ldots) \sim \langle \exp \left( -R'_f(\zeta_1, \zeta_2, \ldots, ) \right) \rangle \text{summed over emissions in blue band} \)
  - observable-dependent — this part done by Monte Carlo (pure \( \alpha_s^n L^n \))
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\( C_{\ell} = \) colour factor; \( q_{\ell} = \) PDF \( r_{\ell}(L) \Rightarrow \alpha_s^n L^{n+1} \); \( r'_{\ell}(L), T(L) \Rightarrow \alpha_s^n L^n \)

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\( S(T(L/a)) = \) large-angle logarithms (proc. dep.) Botts-Kidonakis-Oderda-Sterman '89–'98; Bonciani et al '03

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QCD beyond fixed order (p. 37)

Resummation done systematically

Practice

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### Non-trivial parts:

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- **\(F(\ldots)\)** ~ \(\langle \exp \left( -R' f(\xi_1, \xi_2, \ldots) \right) \rangle\) summed over emissions in blue band
  observable-dependent — this part done by Monte Carlo (pure \(\alpha_s^n L^n\))
QCD beyond fixed order (p. 38)
Resummation done systematically
Practice

CAESAR flow chart

Computer Automated Expert Semi-Analytical Resummer
Banfi, GPS, Zanderighi ’03–’05

START

User supplies observable and Born momenta

Determination of leg properties \( \alpha, \beta, \delta, \gamma \) \([\text{eq.}(3.1)]\)

success

Determination of sufficiently soft and collinear region for subsequent steps

Continuous global? \([a_1 = \ldots = a_n \text{ and eqs.}(3.2)]\)

yes

\( r \text{IRC safe?} \) \([\text{eqs.}(3.4,3.5)]\)

yes

failure of resummation

FAILURE

yes

Additive? \([\text{eq.}(4.1)]\)

no

yes

Establish integration range \((\epsilon)\) for \( \mathcal{F}_2 \) and \( \mathcal{F} \)

Determine zeroes and study their properties (used in computation of \( \mathcal{F} \))

Event-shape like? \([\text{eq.}(3.11)]\)

no

yes

\( \mathcal{F} \) calculable in double precision?

no

yes

\( \mathcal{F} \) and \( \mathcal{F}_2 \) known analytically \([\text{eqs.}(3.26,A.10)]\)

Calculate \( \mathcal{F} \) and \( \mathcal{F}_2 \) in double precision \([\text{eqs.}(3.12,A.9)]\)

success: NLL resummed result

Calculate \( \mathcal{F} \) and \( \mathcal{F}_2 \) in multiple precision \([\text{eqs.}(3.9,A.9)]\)
What it doesn’t do

- Observables that vanish other than through suppression of radiation (e.g. Vector Boson $p_t$ spectrum) have divergence in $g_2(\alpha_s L)$ beyond fixed value of $\alpha_s L$. Rakow & Webber '81; Dasgupta & GPS '02

- For very-inclusive 2-jet cases analytical resummations are in any case more accurate (NNLL) Higgs $p_t$: Bozzi et al '03–05
  Back-to-back EEC: de Florian & Grazzini '04

- For less-inclusive cases, this problem is sometimes ‘academic’ (in region of vanishing X-section).

- Non-global observables are beyond its scope (but perhaps could be included in future).
  - Individual jet properties, or subsets of jets
  - Gap resummations Appleby, Banfi, C. Berger, Dasgupta, Forshaw Kucs, Kyrieleis, Oderda, Seymour, Sterman, …

- Threshold resummations not yet thought about in this framework.
Reproduced/verified all known analytical global resummations

Except for 1 case where it replaces an incomplete result

\[ y_{3D} \]: widely used in fits to \( \alpha_s \)
Banfi, GPS & Zanderighi '01

Correctly identifies cases where it is not able to give correct answer.

New multi-jet resummations in \( e^+e^- \) and DIS

First event-shape resummation for hadron-hadron dijet events

Uses soft-logarithms from Stony Brook group

All results available at http://qcd-caesar.org

Program available on request
Some hadron-collider dijet observables

<table>
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<tr>
<th>Event-shape</th>
<th>Impact of $\eta_{\text{max}}$</th>
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The study of such a wide range of observables would have been nearly impossible without automation...
Normal QCD perturbation theory relies on *infrared & collinear safety* of observable to allow one to restrict matrix elements for $N^pLO$ calculation to $n_{Born} + p$ partons.

In certain (exclusive) regions of phase-space, while formally ($\alpha_s \rightarrow 0$) OK, this is practically insufficient: need *all-order resummation* of logarithmically enhanced terms.

New condition: *recursive infrared and collinear safety*, ensures (together with globalness, coherence) that, for NLL resummed accuracy, it is safe to approximate $n$-parton soft-collinear matrix-element as independent emission.

Enables automation of resummation $\rightarrow$ CAESAR!

First hadron-hadron dijet event-shape resummations

*Many questions for future*. Can the automated resummation be made practical beyond NLL accuracy? Are there issues with coherence in processes with incoming hadrons?
EXTRA SLIDES
Contradiction?

Theoretical calculations are for global observables. But experiments only have detectors in limited rapidity range. (Strictly: series of sub-detectors, of worsening quality as rapidity increases)

Model by cut around beam $|\eta| < \eta_{\text{max}}$

Problems with globalness

Take cut as being edge of most forward detector with momentum or energy resolution:

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Most of cross section may be above that limit — rapidity cut irrelevant. 

Banfi et al. '01

Alternative

Measure just centrally & add recoil term (indirect sensitivity to rest of event):

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R_{\perp,C} \equiv \frac{1}{Q_{\perp,C}} \left| \sum_{i \in C} \vec{q}_{\perp,i} \right|
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Here \( g_2(\alpha_s L) \) diverges for \( L \sim 1/\alpha_s \) (due to cancellations in vector sum) — study distribution only before divergence.
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Grey entries are definitely subject to uncertainty

Note complementarity between observables
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