# QCD at hadron colliders
# Lecture 1: Introduction

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# QUANTUM CHROMODYNAMICS

The theory of quarks, gluons and their interactions

It's central to all modern colliders.
(And QCD is what we're made of)

# The ingredients of QCD

- Quarks (and anti-quarks): they come in 3 colours
- Gluons: a bit like photons in QED

  But there are 8 of them, and they're colour charged
- And a coupling, $\alpha_s$, that's not so small and runs fast

  At LHC, in the range 0.08(@ 5 TeV) to $\mathcal{O}(1)$(@ 0.5 GeV)

# Aims of this course

**I'll try to give you a feel for:**

How QCD works

How theorists handle QCD at high-energy colliders

How experimenters can work with QCD at high-energy colliders

# Lagrangian + colour

Quarks — 3 colours: $\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$

Quark part of Lagrangian:

**Let's write down QCD in full detail**

(There's a lot to absorb here — but it should become more palatable as we return to individual elements later)

$SU(3$

corre

A representation is: $t^A = \frac{1}{2}\lambda^A$,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

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Quark part of Lagrangian:

$$\mathcal{L}_q = \bar{\psi}_a (i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t^C_{ab} \mathcal{A}^C_\mu - m)\psi_b$$

$SU(3)$ local gauge symmetry $\leftrightarrow$ 8 $(= 3^2 - 1)$ generators $t^1_{ab} \dots t^8_{ab}$ corresponding to 8 gluons $\mathcal{A}^1_\mu \dots \mathcal{A}^8_\mu$.

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# Lagrangian: gluonic part

Field tensor: $F^A_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\nu - g_s f_{ABC} \mathcal{A}^B_\mu \mathcal{A}^C_\nu$ $\qquad [t^A, t^B] = i f_{ABC} t^C$

$f_{ABC}$ are structure constants of $SU(3)$ (antisymmetric in all indices — $SU(2)$ equivalent was $\epsilon^{ABC}$). Needed for gauge invariance of gluon part of Lagrangian:

$$\mathcal{L}_G = -\frac{1}{4} F^{\mu\nu}_A F^{A\,\mu\nu}$$

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# Solving QCD

**Two main approaches to solving it**

- Numerical solution with discretized space time (lattice)
- Perturbation theory: assumption that coupling is small

Also: effective theories

# Lattice QCD

- Put all the quark and gluon fields of QCD on a 4D-lattice

  NB: with imaginary time

- Figure out which field configurations are most likely (by Monte Carlo sampling).

- You've solved QCD

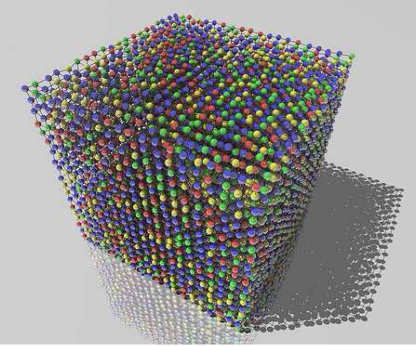
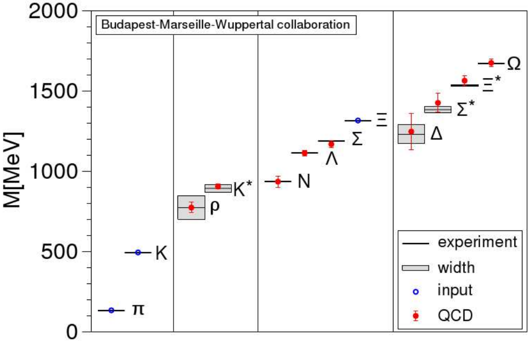


image credits: fdecomite [Flickr]

# Lattice hadron masses

Lattice QCD is great at calculation static properties of a single hadron.

E.g. the hadron mass spectrum




Durr et al ’08

# Lattice for LHC?

How big a lattice do you need for an LHC collision @ 14 TeV?

Lattice spacing: $\dfrac{1}{14\text{ TeV}} \sim 10^{-5}\,\text{fm}$

Lattice extent:

- non-perturbative dynamics for quark/hadron near rest takes place on timescale $t \sim \dfrac{1}{0.5\text{ GeV}} \sim 0.4\text{ fm}/c$
- But quarks at LHC have effective boost factor $\sim 10^4$
- So lattice extent should be $\sim 4000$ fm

Total: need $\sim 4 \times 10^8$ lattice units in each direction, or $3 \times 10^{34}$ nodes total.

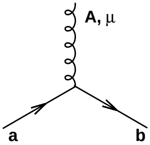
Plus clever tricks to deal with high particle multiplicity, imaginary v. real time, etc.

# Perturbation theory

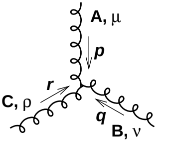
Relies on idea of order-by-order expansion small coupling, $\alpha_s \ll 1$

$$\alpha_s + \underbrace{\alpha_s^2}_{\text{small}} + \underbrace{\alpha_s^3}_{\text{smaller}} + \underbrace{\ldots}_{\text{negligible?}}$$

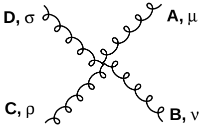
Interaction vertices of Feynman rules:




$-ig_s t^A_{ba} \gamma^\mu$



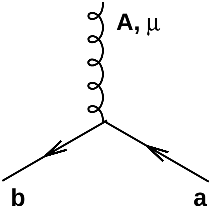

$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$



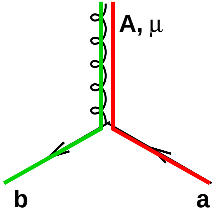

$-ig_s^2 f^{XAC} f^{XBD}[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$

These expressions are fairly complex, so you really don't want to have to deal with too many orders of them!
i.e. $\alpha_s$ had better be small...

# What do Feynman rules mean physically?



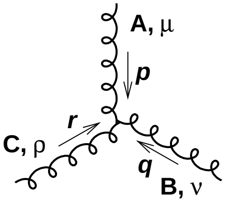

$$\bar{\psi}_b(-ig_s t^A_{ba}\gamma^\mu)\psi_a$$



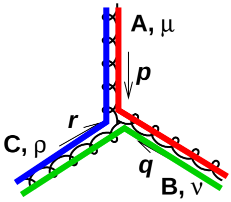

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{\bar{\psi}_b}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{t^1_{ab}}\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\psi_a}$$

A gluon emission **repaints** the quark colour.
A gluon itself carries colour and anti-colour.

# What does "ggg" Feynman rule mean?




$$-g_s f^{ABC}[(p-q)^{\rho} g^{\mu\nu} + (q-r)^{\mu} g^{\nu\rho} + (r-p)^{\nu} g^{\rho\mu}]$$

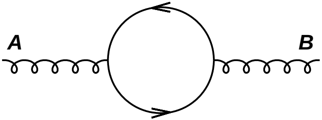



A gluon emission also repaints the gluon colours.

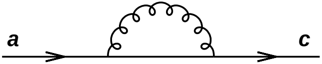
Because a gluon carries colour + anti-colour, it emits $\sim$ twice as strongly as a quark (just has colour)

# Quick guide to colour algebra

$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}, \quad T_R = \tfrac{1}{2}$$




$$\textstyle\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}, \quad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$

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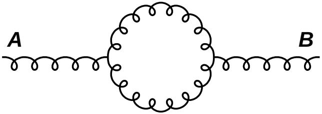
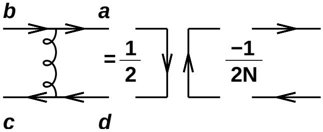



$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_c}\delta_{ab}\delta_{cd} \quad \text{(Fierz)}$$

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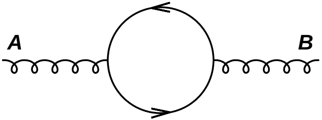


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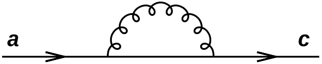


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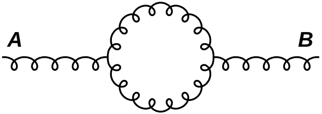


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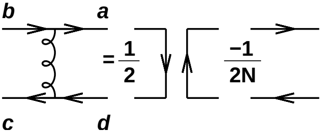


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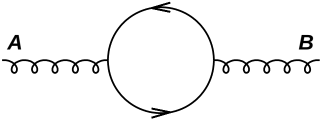


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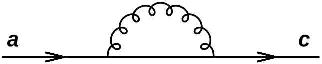


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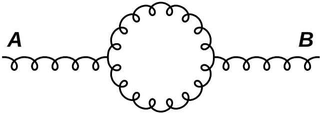


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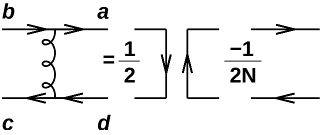


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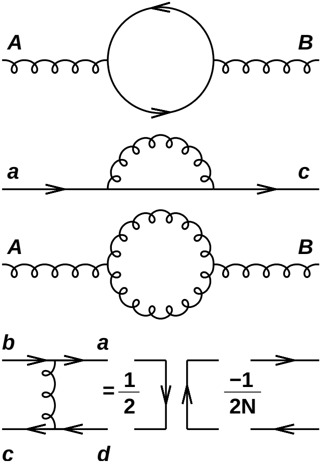


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# How big is the coupling?

All couplings run (QED, QCD, EW), i.e. they depend on the momentum scale ($Q^2$) of your process.

The QCD coupling, $\alpha_s(Q^2)$, runs **fast:**

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s)\,, \qquad \beta(\alpha_s) = -\alpha_s^2(b_0 + b_1\alpha_s + b_2\alpha_s^2 + \ldots)\,,$$

$$b_0 = \frac{11C_A - 2n_f}{12\pi}\,, \qquad b_1 = \frac{17C_A^2 - 5C_A n_f - 3C_F n_f}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2}$$

Note sign: Asymptotic Freedom, due to gluon to self-interaction

2004 Novel prize: Gross, Politzer & Wilczek

- At high scales $Q$, coupling becomes small

  ➥quarks and gluons are almost free, interactions are weak

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$\Lambda \simeq 0.2$ GeV (aka $\Lambda_{QCD}$) is the fundamental scale of QCD, at which coupling blows up.

- $\Lambda$ sets the scale for hadron masses (NB: $\Lambda$ not unambiguously defined wrt higher orders)
- Perturbative calculations valid for scales $Q \gg \Lambda$.

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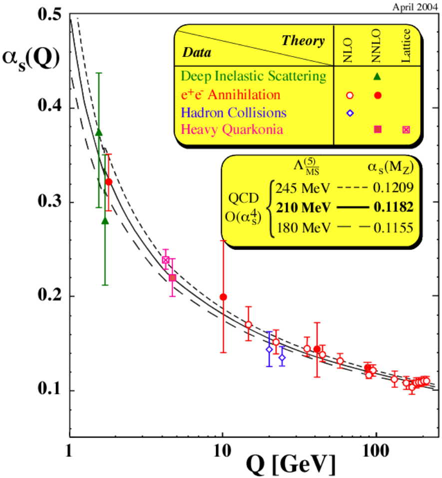
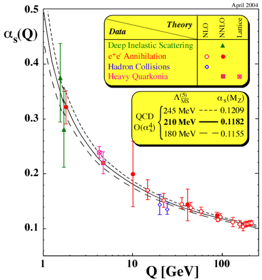


# QCD perturbation theory (PT) & LHC?




- The "new physics" at colliders is searched for at scales $Q \sim p_t \sim 50$ GeV – 5 TeV

  The coupling certainly is small there!

- But we're colliding protons, $m_p \simeq 0.94$ GeV

  The coupling is large!

When we look at QCD events (this one is interpreted as $e^+e^- \to Z \to q\bar{q}$), we see:

- hadrons (PT doesn't hold for them)
- lots of them — so we can't say 1 quark/gluon $\sim$ 1 hadron, and we limit ourselves to 1 or 2 orders of PT.

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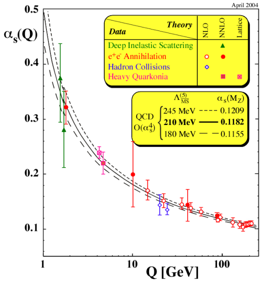


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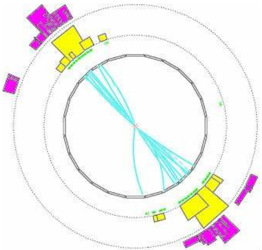

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**Neither lattice QCD nor perturbative QCD can offer a full solution to using QCD at colliders**

What the community has settled on is perturbative QCD inputs + non-perturbative *modelling/factorisation*

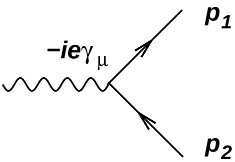
*These lectures:*

- Examine how perturbation theory allows us to understand why QCD events look the way they do.
- Look at the methods available to carry out QCD predictions at hadron colliders
- Discuss how knowledge of QCD can help us search for new physics

# Soft gluon amplitude

Start with $\gamma^* \to q\bar{q}$:

$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2)$$




Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$

Make gluon *soft* $\equiv k \ll p_{1,2}$; ignore terms suppressed by powers of $k$:

$$\mathcal{M}_{q\bar{q}g} \simeq \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left( \frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

$\not{p} v(p) = 0,$
$\not{p}\not{k} + \not{k}\not{p} = 2p.k$

# Soft gluon amplitude

Start with $\gamma^* \to q\bar{q}$:

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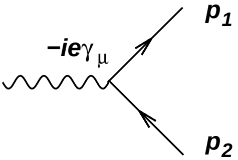


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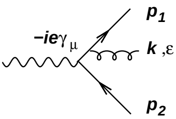


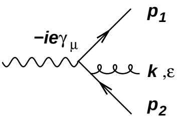


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# Soft gluon amplitude

Start with $\gamma^* \to q\bar{q}$:

$$\bar{u}(p_1) i g_s \not\epsilon t^A \frac{i}{\not p_1 + \not k} i e_q \gamma_\mu v(p_2) = -i g_s \bar{u}(p_1) \not\epsilon \frac{\not p_1 + \not k}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\not A \not B = 2A.B - \not B \not A$:

$$= -i g_s \bar{u}(p_1)[2\epsilon.(p_1+k) - (\not p_1 + \not k)\not\epsilon] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\bar{u}(p_1)\not p_1 = 0$ and $k \ll p_1$ ($p_1, k$ massless)

$$\simeq -i g_s \bar{u}(p_1)[2\epsilon.p_1] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

$$= -i g_s \frac{p_1.\epsilon}{p_1.k} \underbrace{\bar{u}(p_1) e_q \gamma_\mu t^A v(p_2)}_{\text{pure QED spinor structure}}$$

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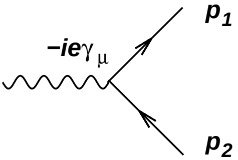


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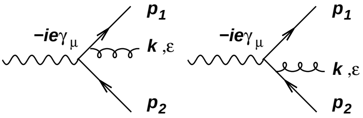


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# Squared amplitude

$$|M^2_{q\bar{q}g}| \simeq \sum_{A,\text{pol}} \left| \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left( \frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

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Include phase space:

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Note property of factorisation into hard $q\bar{q}$ piece and soft-gluon emission piece, $dS$.

$$dS = EdE\, d\cos\theta\, \frac{d\phi}{2\pi} \cdot \frac{2\alpha_s C_F}{\pi} \frac{2p_1.p_2}{(2p_1.k)(2p_2.k)}$$

$\theta = \theta_{p_1 k}$
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# Soft & collinear gluon emission

Take squared matrix element and rewrite in terms of $E$, $\theta$,

$$\frac{2p_1.p_2}{(2p_1.k)(2p_2.k)} = \frac{1}{E^2(1-\cos^2\theta)}$$

So final expression for soft gluon emission is

$$d\mathcal{S} = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

NB:

- It *diverges* for $E \to 0$ — *infrared (or soft) divergence*
- It *diverges* for $\theta \to 0$ and $\theta \to \pi$ — *collinear divergence*

Soft, collinear divergences derived here in specific context of $e^+e^- \to q\bar{q}$
**But they are a very general property of QCD**

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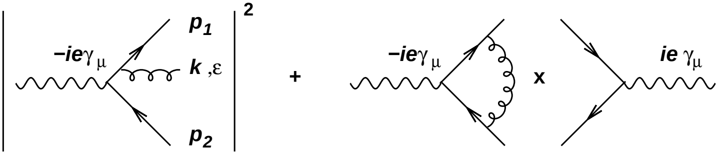
# Real-virtual cancellations: total X-sctn

Total cross section: sum of all real and virtual diagrams




Total cross section must be *finite*. If real part has divergent integration, so must virtual part. (Unitarity, conservation of probability)

$$\sigma_{tot} = \sigma_{q\bar{q}} \left( 1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} R(E/Q, \theta) - \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} V(E/Q, \theta) \right)$$

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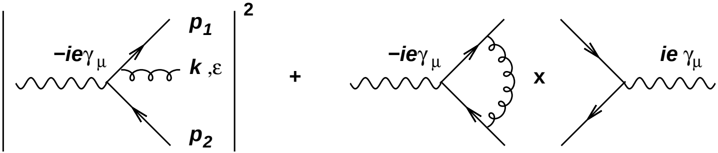


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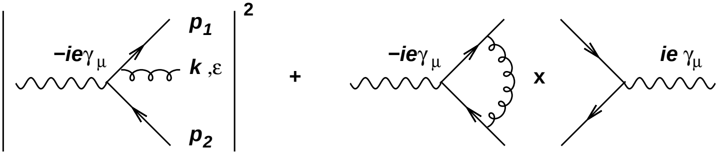


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$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} \left(R(E/Q,\theta) - V(E/Q,\theta)\right)\right)$$

- From calculation: $\lim_{E\to 0} R(E/Q,\theta) = 1$.
- For every divergence $R(E/Q,\theta)$ and $V(E/Q,\theta)$ should cancel:

$$\lim_{E\to 0}(R - V) = 0\,, \qquad \lim_{\theta\to 0,\pi}(R - V) = 0$$

Result:

- corrections to $\sigma_{tot}$ come from hard ($E \sim Q$), large-angle gluons
- Soft gluons don't matter:
  - Physics reason: soft gluons emitted on long timescale $\sim 1/(E\theta^2)$ relative to collision ($1/Q$) — cannot influence cross section.
  - Transition to hadrons also occurs on long time scale ($\sim 1/\Lambda$) — and can also be ignored.
- Correct renorm. scale for $\alpha_s$: $\mu \sim Q$ — perturbation theory valid.

# total X-section (cont.)

Dependence of total cross section on only *hard* gluons is reflected in 'good behaviour' of perturbation series:

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + 1.045\frac{\alpha_s(Q)}{\pi} + 0.94\left(\frac{\alpha_s(Q)}{\pi}\right)^2 - 15\left(\frac{\alpha_s(Q)}{\pi}\right)^3 + \cdots\right)$$

(Coefficients given for $Q = M_Z$)