Phenomenology

Gavin P. Salam

LPTHE, Universities of Paris VI and VII and CNRS

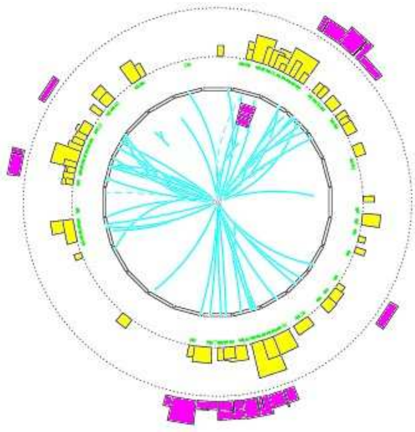
BUSSTEPP
Plymouth, August 2004

Phenomenology

Lecture 4

(QCD jets)

Understanding jets



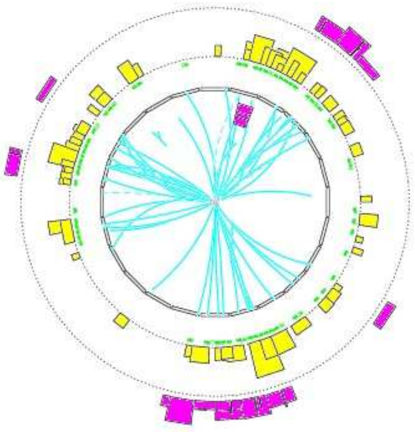
Previous lecture

- Divergent matrix element for emission of soft and collinear gluons.
- ‘Good’ observables are insensitive to this — infrared and collinear safe.
- But complex event structure is still present (and must be understood for many practical uses of QCD).

This lecture

- Try to see how event structure builds up.
- See when that information is relevant

Understanding jets



Previous lecture

- Divergent matrix element for emission of soft and collinear gluons.
- 'Good' observables are insensitive to this — infrared and collinear safe.
- But complex event structure is still present (and must be understood for many practical uses of QCD).

This lecture

- Try to see how event structure builds up.
- See when that information is relevant

Outline

'Technical' aim of this lecture is to introduce you to methods for calculations at arbitrarily high orders of perturbation theory.

- Can't be done exactly

Outline

'Technical' aim of this lecture is to introduce you to methods for calculations at arbitrarily high orders of perturbation theory.

- Can't be done exactly
  - ➥ Perturbation theory too cumbersome, *e.g.* $n!$ growth of number of diags.
  - ➥ Current 'exact' state of the art is slowly evolving from NLO calculations to NNLO.
- exploit presence of an addition small parameter over and above the coupling

Outline

'Technical' aim of this lecture is to introduce you to methods for calculations at arbitrarily high orders of perturbation theory.

- Can't be done exactly
  - ➥ Perturbation theory too cumbersome, *e.g.* $n!$ growth of number of diags.
  - ➥ Current 'exact' state of the art is slowly evolving from NLO calculations to NNLO.
- exploit presence of an addition small parameter over and above the coupling

Outline

'Technical' aim of this lecture is to introduce you to methods for calculations at arbitrarily high orders of perturbation theory.

- Can't be done exactly
  - ➥ Perturbation theory too cumbersome, *e.g.* $n!$ growth of number of diags.
  - ➥ Current 'exact' state of the art is slowly evolving from NLO calculations to NNLO.
- exploit presence of an addition small parameter over and above the coupling
  - ➥ softness of emissions

Outline

'Technical' aim of this lecture is to introduce you to methods for calculations at arbitrarily high orders of perturbation theory.

- Can't be done exactly
  - ➥ Perturbation theory too cumbersome, *e.g.* $n!$ growth of number of diags.
  - ➥ Current 'exact' state of the art is slowly evolving from NLO calculations to NNLO.
- exploit presence of an addition small parameter over and above the coupling
  - ➥ softness of emissions

Outline

'Technical' aim of this lecture is to introduce you to methods for calculations at arbitrarily high orders of perturbation theory.

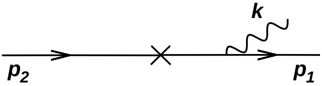
- Can't be done exactly
  - ➥ Perturbation theory too cumbersome, *e.g.* $n!$ growth of number of diags.
  - ➥ Current 'exact' state of the art is slowly evolving from NLO calculations to NNLO.
- exploit presence of an addition small parameter over and above the coupling
  - ➥ softness of emissions

QED

To understand jet structure need to consider *multiple soft gluon emission.*

Before doing so, useful to examine some simple QED cases.

Soft photon radiated from e^+e^- pair



$$M_{e^+e^-\gamma} = -M_{e^+e^-} g_e \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

Divergent denominator: near on-shell propagator

This is the source of the enhancement of soft-photon radiation.

Same mechanism in QCD

Squared amplitude for photon emission:

$$g_e^2 \frac{2p_1.p_2}{p_1.k\, p_2.k}$$

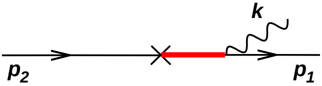
Now consider two extensions of this.

QED

To understand jet structure need to consider *multiple soft gluon emission.*

Before doing so, useful to examine some simple QED cases.

Soft photon radiated from e^+e^- pair



$$M_{e^+e^-\gamma} = -M_{e^+e^-} g_e \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

Divergent denominator: near on-shell propagator

This is the source of the enhancement of soft-photon radiation.

Same mechanism in QCD

Squared amplitude for photon emission:

$$g_e^2 \frac{2p_1.p_2}{p_1.k\, p_2.k}$$

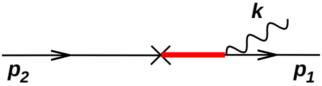
Now consider two extensions of this.

QED

To understand jet structure need to consider *multiple soft gluon emission.*

Before doing so, useful to examine some simple QED cases.

Soft photon radiated from e^+e^- pair



$$M_{e^+e^-\gamma} = -M_{e^+e^-} g_e \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

Divergent denominator: near on-shell propagator

This is the source of the enhancement of soft-photon radiation.

Same mechanism in QCD

Squared amplitude for photon emission:

$$g_e^2 \frac{2p_1.p_2}{p_1.k\, p_2.k}$$

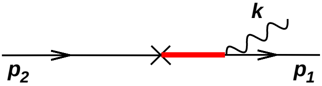
Now consider two extensions of this.

QED

To understand jet structure need to consider *multiple soft gluon emission.*

Before doing so, useful to examine some simple QED cases.

Soft photon radiated from e^+e^- pair



$$M_{e^+e^-\gamma} = -M_{e^+e^-} g_e \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

Divergent denominator: near on-shell propagator

This is the source of the enhancement of soft-photon radiation.

Same mechanism in QCD

Squared amplitude for photon emission:

$$g_e^2 \frac{2p_1.p_2}{p_1.k \, p_2.k}$$

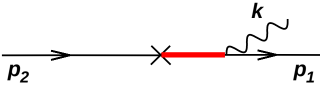
Now consider two extensions of this.

QED

To understand jet structure need to consider *multiple soft gluon emission.*

Before doing so, useful to examine some simple QED cases.

Soft photon radiated from e^+e^- pair



$$M_{e^+e^-\gamma} = -M_{e^+e^-} g_e \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

Divergent denominator: near on-shell propagator

This is the source of the enhancement of soft-photon radiation.

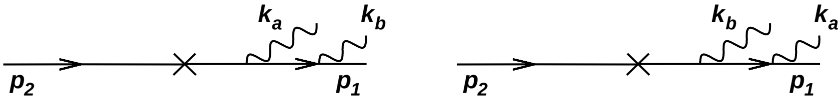
Same mechanism in QCD

Squared amplitude for photon emission:

$$g_e^2 \frac{2p_1.p_2}{p_1.k\, p_2.k}$$

Now consider two extensions of this.

Ordered two-photon emission



k_a harder than k_b ($k_a.p_1 \gg k_b.p$)

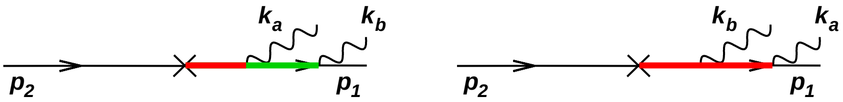
$$\boxed{g_e^2 \frac{p_1.\epsilon_a}{p_1.k_a}\frac{p_1.\epsilon_b}{p_1.k_b}} \gg g_e^2 \frac{p_1.\epsilon_b}{p_1.k_a}\frac{p_1.\epsilon_a}{p_1.k_a}$$

k_a softer than k_b ($k_a.p_1 \ll k_b.p$)

$$g_e^2 \frac{p_1.\epsilon_a}{p_1.k_b}\frac{p_1.\epsilon_b}{p_1.k_b} \ll \boxed{g_e^2 \frac{p_1.\epsilon_b}{p_1.k_b}\frac{p_1.\epsilon_a}{p_1.k_a}}$$

Softer photon always emitted second (cannot be emitted from a more off-shell line). *Result looks like independent emission of the two photons.*

Ordered two-photon emission



k_a harder than k_b ($k_a.p_1 \gg k_b.p$)

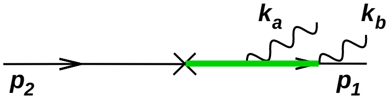
$$\boxed{g_e^2 \frac{p_1.\epsilon_a}{p_1.k_a} \frac{p_1.\epsilon_b}{p_1.k_b}} \gg g_e^2 \frac{p_1.\epsilon_b}{p_1.k_a} \frac{p_1.\epsilon_a}{p_1.k_a}$$

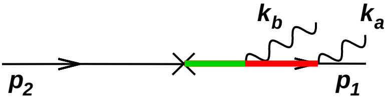
k_a softer than k_b ($k_a.p_1 \ll k_b.p$)

$$g_e^2 \frac{p_1.\epsilon_a}{p_1.k_b} \frac{p_1.\epsilon_b}{p_1.k_b} \ll \boxed{g_e^2 \frac{p_1.\epsilon_b}{p_1.k_b} \frac{p_1.\epsilon_a}{p_1.k_a}}$$

Softer photon always emitted second (cannot be emitted from a more off-shell line). *Result looks like independent emission of the two photons.*

Ordered two-photon emission





k_a harder than k_b ($k_a.p_1 \gg k_b.p$)

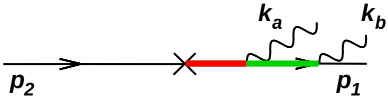
$$\boxed{g_e^2 \frac{p_1.\epsilon_a}{p_1.k_a} \frac{p_1.\epsilon_b}{p_1.k_b}} \gg g_e^2 \frac{p_1.\epsilon_b}{p_1.k_a} \frac{p_1.\epsilon_a}{p_1.k_a}$$

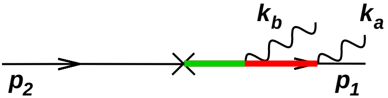
k_a softer than k_b ($k_a.p_1 \ll k_b.p$)

$$g_e^2 \frac{p_1.\epsilon_a}{p_1.k_b} \frac{p_1.\epsilon_b}{p_1.k_b} \ll \boxed{g_e^2 \frac{p_1.\epsilon_b}{p_1.k_b} \frac{p_1.\epsilon_a}{p_1.k_a}}$$

Softer photon always emitted second (cannot be emitted from a more off-shell line). *Result looks like independent emission of the two photons.*

Ordered two-photon emission





k_a harder than k_b $(k_a.p_1 \gg k_b.p)$

$$\boxed{g_e^2 \frac{p_1.\epsilon_a}{p_1.k_a} \frac{p_1.\epsilon_b}{p_1.k_b}} \gg g_e^2 \frac{p_1.\epsilon_b}{p_1.k_a} \frac{p_1.\epsilon_a}{p_1.k_a}$$

k_a softer than k_b $(k_a.p_1 \ll k_b.p)$

$$g_e^2 \frac{p_1.\epsilon_a}{p_1.k_b} \frac{p_1.\epsilon_b}{p_1.k_b} \ll \boxed{g_e^2 \frac{p_1.\epsilon_b}{p_1.k_b} \frac{p_1.\epsilon_a}{p_1.k_a}}$$

Softer photon always emitted second (cannot be emitted from a more off-shell line). *Result looks like independent emission of the two photons.*

Ordered two-photon emission (cont.)

Squared amplitude for double soft photon emission

$$|M_{e^+e^-\gamma\gamma}|^2 = |M_{e^+e^-}|^2\, g_e^4 \frac{2p_1.p_2}{p_1.k_a\, p_2.k_a}\, \frac{2p_1.p_2}{p_1.k_b\, p_2.k_b} \quad (1)$$

$$= |M_{e^+e^-a}|^2 g_e^2 \frac{2p_1.p_2}{p_1.k_b\, p_2.k_b} \quad (2)$$

Independent emission, eq.(1), holds if both are soft.

Now suppose k_a is hard, k_b soft

- No divergent propagators for k_a
- k_b radiated 'after' k_a
- factorisation, eq.(2), still holds
- as long as p_1, p_2 are the e^+e^- momenta *after* emission of k_a.

[& must use full $|M_{e^+e^-\gamma_a}|^2$]

Ordered two-photon emission (cont.)

Squared amplitude for double soft photon emission

$$|M_{e^+e^-\gamma\gamma}|^2 = |M_{e^+e^-}|^2 \, g_e^4 \frac{2p_1.p_2}{p_1.k_a \, p_2.k_a} \, \frac{2p_1.p_2}{p_1.k_b \, p_2.k_b} \quad (1)$$

$$= |M_{e^+e^-a}|^2 g_e^2 \frac{2p_1.p_2}{p_1.k_b \, p_2.k_b} \quad (2)$$

Independent emission, eq.(1), holds if both are soft.

Now suppose k_a is hard, k_b soft

- No divergent propagators for k_a
- k_b radiated ‘after’ k_a
- factorisation, eq.(2), still holds
- as long as p_1, p_2 are the e^+e^- momenta *after* emission of k_a.

[& must use full $|M_{e^+e^-\gamma_a}|^2$]

Ordered two-photon emission (cont.)

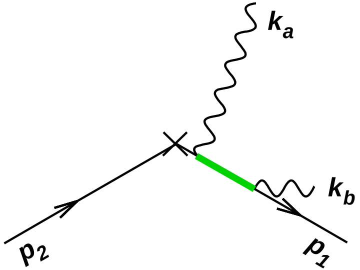
Squared amplitude for double soft photon emission

$$|M_{e^+e^-\gamma\gamma}|^2 = |M_{e^+e^-}|^2 \, g_e^4 \frac{2p_1.p_2}{p_1.k_a \, p_2.k_a} \, \frac{2p_1.p_2}{p_1.k_b \, p_2.k_b} \quad (1)$$

$$= |M_{e^+e^-a}|^2 g_e^2 \frac{2p_1.p_2}{p_1.k_b \, p_2.k_b} \quad (2)$$

Independent emission, eq.(1), holds if both are soft.

Now suppose k_a is hard, k_b soft



- No divergent propagators for k_a
- k_b radiated 'after' k_a
- factorisation, eq.(2), still holds
- as long as p_1, p_2 are the e^+e^- momenta *after* emission of k_a.

[& must use full $|M_{e^+e^-\gamma_a}|^2$]

Ordered two-photon emission (cont.)

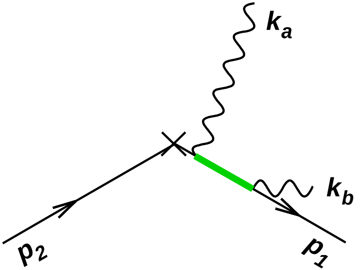
Squared amplitude for double soft photon emission

$$|M_{e^+e^-\gamma\gamma}|^2 = |M_{e^+e^-}|^2 \, g_e^4 \frac{2p_1.p_2}{p_1.k_a \, p_2.k_a} \, \frac{2p_1.p_2}{p_1.k_b \, p_2.k_b} \quad (1)$$

$$= |M_{e^+e^-a}|^2 g_e^2 \frac{2p_1.p_2}{p_1.k_b \, p_2.k_b} \quad (2)$$

Independent emission, eq.(1), holds if both are soft.

<u>Now suppose k_a is hard, k_b soft</u>



- No divergent propagators for k_a
- k_b radiated 'after' k_a
- factorisation, eq.(2), still holds
- as long as p_1, p_2 are the e^+e^- momenta *after* emission of k_a.

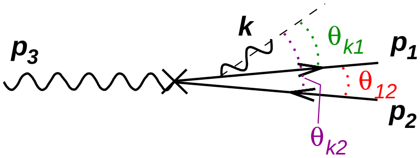
[& must use full $|M_{e^+e^-\gamma_a}|^2$]

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.

What is radiation pattern of soft photon k?



$$g_e^2 \frac{2p_1.p_2}{p_1.k\; p_2.k} = \frac{g_e^2}{\omega_k^2} \frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

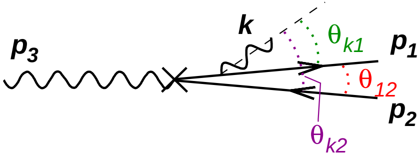
Two ‘ordered’ cases:

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.

What is radiation pattern of soft photon k?



$$g_e^2 \frac{2p_1.p_2}{p_1.k\; p_2.k} = \frac{g_e^2}{\omega_k^2} \frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

Two 'ordered' cases:

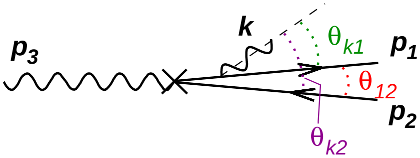
$\theta_{k1} \ll \theta_{k2} \simeq \theta_{12} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k1}^2}$ Close to e^- or e^+, usual $1/\theta^2$ radiation pattern

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.

What is radiation pattern of soft photon k?



$$g_e^2 \frac{2p_1.p_2}{p_1.k\, p_2.k} = \frac{g_e^2}{\omega_k^2} \frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

Two 'ordered' cases:

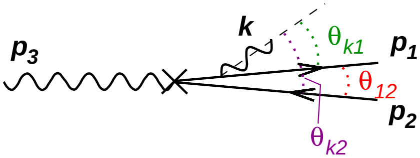
$\theta_{k1} \ll \theta_{k2} \simeq \theta_{12} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k1}^2}$ Close to e^- or e^+, usual $1/\theta^2$ radiation pattern.

$\theta_{12} \ll \theta_{k1} \simeq \theta_{k2} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{\theta_{12}^2}{\theta_{k1}^4}$ At large angles from e^+e^- pair, do not see their charge (overall neutral) — *photon radiation suppressed.*

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.



What is radiation pattern of soft photon k?

$$g_e^2 \frac{2p_1.p_2}{p_1.k\; p_2.k} = \frac{g_e^2}{\omega_k^2} \frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

Two 'ordered' cases:

$\theta_{k1} \ll \theta_{k2} \simeq \theta_{12} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k1}^2}$ Close to e^- or e^+, usual $1/\theta^2$ radiation pattern.

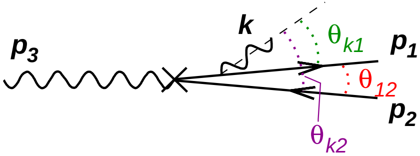
$\theta_{12} \ll \theta_{k1} \simeq \theta_{k2} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{\theta_{12}^2}{\theta_{k1}^4}$ At large angles from e^+e^- pair, do not see their charge (overall neutral) — *photon radiation suppressed.* This is COHERENCE

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.

What is radiation pattern of soft photon k?



$$g_e^2 \frac{2p_1.p_2}{p_1.k\; p_2.k} = \frac{g_e^2}{\omega_k^2}\frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

Two 'ordered' cases:

$\theta_{k2} \ll \theta_{k1} \simeq \theta_{12} \quad \rightarrow \quad \dfrac{4g_e^2}{\omega_k^2}\dfrac{1}{\theta_{k2}^2}$

Close to e^- or e^+, usual $1/\theta^2$ radiation pattern.

$\theta_{12} \ll \theta_{k1} \simeq \theta_{k2} \quad \rightarrow \quad \dfrac{4g_e^2}{\omega_k^2}\dfrac{\theta_{12}^2}{\theta_{k1}^4}$

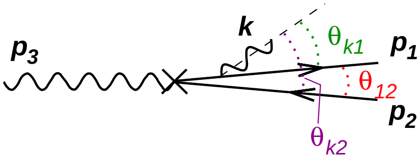
At large angles from e^+e^- pair, do not see their charge (overall neutral) — *photon radiation suppressed.* This is COHERENCE

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.

What is radiation pattern of soft photon k?



$$g_e^2 \frac{2p_1.p_2}{p_1.k\; p_2.k} = \frac{g_e^2}{\omega_k^2} \frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

Two 'ordered' cases:

$\theta_{k2} \ll \theta_{k1} \simeq \theta_{12} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k2}^2}$

Close to e^- or e^+, usual $1/\theta^2$ radiation pattern.

$\theta_{12} \ll \theta_{k1} \simeq \theta_{k2} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{\theta_{12}^2}{\theta_{k1}^4}$

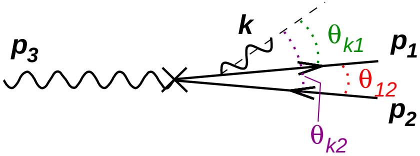
At large angles from e^+e^- pair, do not see their charge (overall neutral) — *photon radiation suppressed.* This is COHERENCE

QED Coherence

Consider rare situation in QED (instructive for later QCD case)

Hard photon p_3, & outgoing e^+e^- pair close in angle, $\theta_{12} \ll 1$.

What is radiation pattern of soft photon k?



$$g_e^2 \frac{2p_1.p_2}{p_1.k\ p_2.k} = \frac{g_e^2}{\omega_k^2} \frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

Two ‘ordered’ cases:

$$\theta_{k2} \ll \theta_{k1} \simeq \theta_{12} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k2}^2}$$

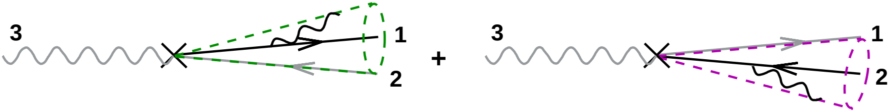
Close to e^- or e^+, usual $1/\theta^2$ radiation pattern.

$$\theta_{12} \ll \theta_{k1} \simeq \theta_{k2} \quad \rightarrow \quad \frac{4g_e^2}{\omega_k^2}\frac{\theta_{12}^2}{\theta_{k1}^4}$$

At large angles from e^+e^- pair, do not see their charge (overall neutral) — *photon radiation suppressed.* This is COHERENCE

QED Coherence (cont.)

Photon emission from e^+e^- pair behaves, approximately, as if it is restricted to two cones of radiation:



$$\frac{g_e^2}{\omega_k^2}\frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})} \simeq$$

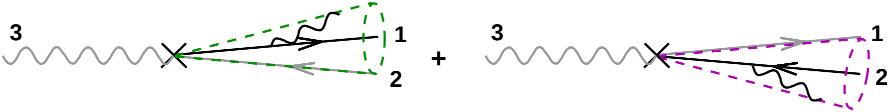
$$\frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k1}^2}\Theta(\theta_{12}-\theta_{k1}) + \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k2}^2}\Theta(\theta_{12}-\theta_{k2})$$

Called *angular ordering*. Relation is exact after integration over angles.

Coherent sum of radiation reduces to *incoherent sum* (over cones)

QED Coherence (cont.)

Photon emission from e^+e^- pair behaves, approximately, as if it is restricted to two cones of radiation:



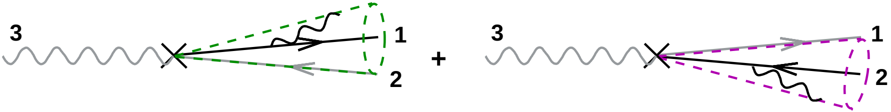
$$\frac{g_e^2}{\omega_k^2}\frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})} \simeq \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k1}^2}\Theta(\theta_{12}-\theta_{k1}) + \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k2}^2}\Theta(\theta_{12}-\theta_{k2})$$

Called *angular ordering*. Relation is exact after integration over angles.

Coherent sum of radiation reduces to *incoherent sum* (over cones)

QED Coherence (cont.)

Photon emission from e^+e^- pair behaves, approximately, as if it is restricted to two cones of radiation:



$$\frac{g_e^2}{\omega_k^2}\frac{2(1-\cos\theta_{12})}{(1-\cos\theta_{k1})(1-\cos\theta_{k2})} \simeq \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k1}^2}\Theta(\theta_{12}-\theta_{k1}) + \frac{4g_e^2}{\omega_k^2}\frac{1}{\theta_{k2}^2}\Theta(\theta_{12}-\theta_{k2})$$

Called *angular ordering*. Relation is exact after integration over angles.

Coherent sum of radiation reduces to *incoherent sum* (over cones)

Extension to QCD

Ideas of multiple soft emission and coherence apply also to QCD.

'Novel' aspects come from $SU(3)$ (colour), in particular gluon's charge.

$k_b \ll k_a \ll p_1, p_2$ (just maximally divergent diags; 'a' radiated off p_2)

[Take $M_{q\bar{q}} = \bar{u}(p_1) \, X \, v(p_2)$]

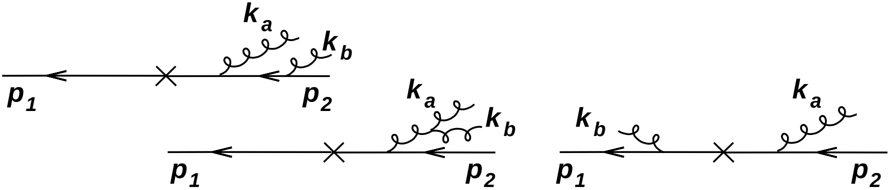
$$M_{q\bar{q}ab} = -g_s^2 \frac{p_2.\epsilon_a}{p_2.k_a} \bar{u}(p_1) \, X \, \left(-t^A t^B \frac{p_2.\epsilon_b}{p_2.k_b} + if^{ABC} t^C \frac{k_a.\epsilon_b}{k_a.k_b} + t^B t^A \frac{p_1.\epsilon_b}{p_1.k_b} \right) v(p_2)$$

Extension to QCD

Ideas of multiple soft emission and coherence apply also to QCD.

'Novel' aspects come from $SU(3)$ (colour), in particular gluon's charge.

$k_b \ll k_a \ll p_1, p_2$ (just maximally divergent diags; 'a' radiated off p_2)



[Take $M_{q\bar{q}} = \bar{u}(p_1)\, X\, v(p_2)$]

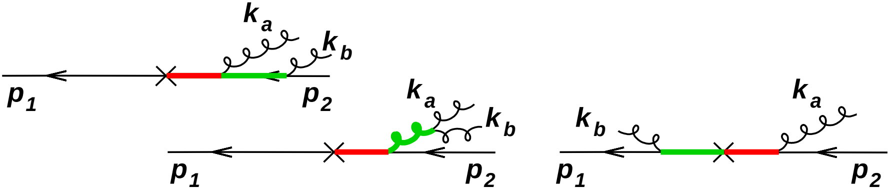
$$M_{q\bar{q}ab} = -g_s^2 \frac{p_2.\epsilon_a}{p_2.k_a} \bar{u}(p_1)\, X \left(-t^A t^B \frac{p_2.\epsilon_b}{p_2.k_b} + if^{ABC} t^C \frac{k_a.\epsilon_b}{k_a.k_b} + t^B t^A \frac{p_1.\epsilon_b}{p_1.k_b} \right) v(p_2)$$

Extension to QCD

Ideas of multiple soft emission and coherence apply also to QCD.

'Novel' aspects come from $SU(3)$ (colour), in particular gluon's charge.

$k_b \ll k_a \ll p_1, p_2$ (just maximally divergent diags; 'a' radiated off p_2)



[Take $M_{q\bar{q}} = \bar{u}(p_1)\,X\,v(p_2)$]

$$M_{q\bar{q}ab} = -g_s^2 \frac{p_2.\epsilon_a}{p_2.k_a} \bar{u}(p_1)\,X\,\left(-t^A t^B \frac{p_2.\epsilon_b}{p_2.k_b} + i f^{ABC} t^C \frac{k_a.\epsilon_b}{k_a.k_b} + t^B t^A \frac{p_1.\epsilon_b}{p_1.k_b}\right) v(p_2)$$

Square the amplitude

Use '(a1)' to mean $p_a.p_1$ etc.

$$
\begin{aligned}
|M_{q\bar{q}ab}|^2 = \frac{|M_{q\bar{q}}|^2}{N_c} 2g_s^4 \frac{(12)}{(1a)(a2)} \Bigg(& \frac{(12)}{(1b)(b2)}\, 2\,\mathrm{Tr}(t^A t^B t^A t^B) \\
& + \frac{(a2)}{(ab)(b2)} i f^{ABC}\, \mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) \\
& - \frac{(a1)}{(ab)(b1)} i f^{ABC}\, \mathrm{Tr}(t^C t^A t^B - t^B t^A t^C) \Bigg)
\end{aligned}
$$

Use

- $t^A t^B t^A = -\frac{1}{2N_c} t^B \to \mathrm{Tr}(t^A t^B t^A t^B) = -\mathrm{Tr}(t^B t^B)/2N_c = -C_F/2$
- $[t^A, t^B] = i f^{ABC} \to i f^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) =$
  $f^{ABC} f^{ABD}\,\mathrm{Tr}(t^C t^D) = C_A\,\mathrm{Tr}(t^C t^C) = C_A^2 C_F$
  (recall $C_A = N_c$)

Square the amplitude

Use '(a1)' to mean $p_a.p_1$ etc.

$$|M_{q\bar{q}ab}|^2 = \frac{|M_{q\bar{q}}|^2}{N_c} 2g_s^4 \frac{(12)}{(1a)(a2)} \left(\frac{(12)}{(1b)(b2)} 2\,\mathrm{Tr}(t^A t^B t^A t^B) + \frac{(a2)}{(ab)(b2)} if^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) - \frac{(a1)}{(ab)(b1)} if^{ABC}\,\mathrm{Tr}(t^C t^A t^B - t^B t^A t^C) \right)$$

Use

- $t^A t^B t^A = -\frac{1}{2N_c} t^B \rightarrow \mathrm{Tr}(t^A t^B t^A t^B) = -\mathrm{Tr}(t^B t^B)/2N_c = -C_F/2$
- $[t^A, t^B] = if^{ABC} \rightarrow if^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) = f^{ABC} f^{ABD}\,\mathrm{Tr}(t^C t^D) = C_A\,\mathrm{Tr}(t^C t^C) = C_A^2 C_F$

(recall $C_A = N_c$)

Square the amplitude

Use '(a1)' to mean $p_a.p_1$ etc.

$$|M_{q\bar{q}ab}|^2 = \frac{|M_{q\bar{q}}|^2}{N_c} 2g_s^4 \frac{(12)}{(1a)(a2)} \left(\frac{(12)}{(1b)(b2)} \, 2\,\mathrm{Tr}(t^A t^B t^A t^B) + \frac{(a2)}{(ab)(b2)} if^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) - \frac{(a1)}{(ab)(b1)} if^{ABC}\,\mathrm{Tr}(t^C t^A t^B - t^B t^A t^C) \right)$$

Use

- $t^A t^B t^A = -\frac{1}{2N_c} t^B \to \mathrm{Tr}(t^A t^B t^A t^B) = -\mathrm{Tr}(t^B t^B)/2N_c = -C_F/2$
- $[t^A, t^B] = if^{ABC} \to if^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) = f^{ABC} f^{ABD}\,\mathrm{Tr}(t^C t^D) = C_A\,\mathrm{Tr}(t^C t^C) = C_A^2 C_F$

(recall $C_A = N_c$)

Square the amplitude

Use '(a1)' to mean $p_a.p_1$ etc.

$$|M_{q\bar{q}ab}|^2 = \frac{|M_{q\bar{q}}|^2}{N_c} 2g_s^4 \frac{(12)}{(1a)(a2)} \left(\frac{(12)}{(1b)(b2)} \, 2\,\mathrm{Tr}(t^A t^B t^A t^B) + \frac{(a2)}{(ab)(b2)} i f^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) - \frac{(a1)}{(ab)(b1)} i f^{ABC}\,\mathrm{Tr}(t^C t^A t^B - t^B t^A t^C) \right)$$

Use

- $t^A t^B t^A = -\frac{1}{2N_c} t^B \to \mathrm{Tr}(t^A t^B t^A t^B) = -\mathrm{Tr}(t^B t^B)/2N_c = -C_F/2$
- $[t^A, t^B] = i f^{ABC} \to i f^{ABC}\,\mathrm{Tr}(t^C t^B t^A - t^A t^B t^C) = f^{ABC} f^{ABD}\,\mathrm{Tr}(t^C t^D) = C_A\,\mathrm{Tr}(t^C t^C) = C_A^2 C_F$

(recall $C_A = N_c$)

Squared amplitude

Write as $|M_{q\bar{q}a}^2| \times$ (emission of b)

$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 2g_s^2 \left(-\frac{1}{2N_c}\frac{(12)}{(1b)(b2)} + \frac{N_c}{2}\frac{(a1)}{(ab)(b1)} + \frac{N_c}{2}\frac{(a2)}{(ab)(b2)} \right)$$

Note structure as *incoherent sum over dipoles*

Consistency: off each quark we have

$$-\frac{1}{2N_c} + \frac{1}{2N_c} = -\frac{1}{2N_c} + \frac{1}{2N_c} = C_F$$

For gluon

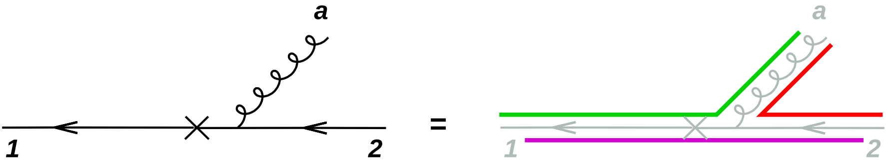
$$\frac{1}{2N_c} + \frac{1}{2N_c} = C_A$$

Squared amplitude

Write as $|M^2_{q\bar{q}a}| \times$ (emission of b)

$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 2g_s^2 \left(-\frac{1}{2N_c}\frac{(12)}{(1b)(b2)} + \frac{N_c}{2}\frac{(a1)}{(ab)(b1)} + \frac{N_c}{2}\frac{(a2)}{(ab)(b2)} \right)$$

Note structure as *incoherent sum over dipoles*



Consistency: off each quark we have

$$-\frac{1}{2N_c} + \frac{1}{2N_c} = -\frac{1}{2N_c} + \frac{1}{2N_c} = C_F$$

For gluon

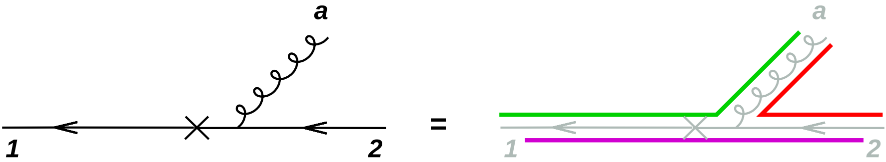
$$\frac{1}{2N_c} + \frac{1}{2N_c} = C_A$$

Squared amplitude

Write as $|M^2_{q\bar{q}a}| \times$ (emission of b)

$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 2g_s^2 \left(-\frac{1}{2N_c}\frac{(12)}{(1b)(b2)} + \frac{N_c}{2}\frac{(a1)}{(ab)(b1)} + \frac{N_c}{2}\frac{(a2)}{(ab)(b2)} \right)$$

Note structure as *incoherent sum over dipoles*



Consistency: off each quark we have

$$-\frac{1}{2N_c} + \frac{1}{2N_c} = -\frac{1}{2N_c} + \frac{1}{2N_c} = C_F$$

For gluon

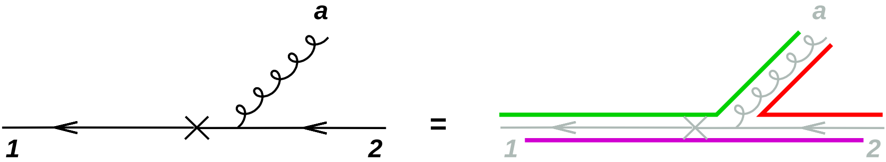
$$\frac{1}{2N_c} + \frac{1}{2N_c} = C_A$$

Squared amplitude

Write as $|M^2_{q\bar{q}a}| \times$ (emission of b)

$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 2g_s^2 \left(-\frac{1}{2N_c}\frac{(12)}{(1b)(b2)} + \frac{N_c}{2}\frac{(a1)}{(ab)(b1)} + \frac{N_c}{2}\frac{(a2)}{(ab)(b2)} \right)$$

Note structure as *incoherent sum over dipoles*



Consistency: off each quark we have

$$-\frac{1}{2N_c} + \frac{1}{2N_c} = -\frac{1}{2N_c} + \frac{1}{2N_c} = C_F$$

For gluon

$$\frac{1}{2N_c} + \frac{1}{2N_c} = C_A$$

Organizing squared amplitude

We have already assumed coupling is small and that gluon energies are *strongly ordered* ($\omega_b \ll \omega_a$).

To proceed towards simple all-orders 'incoherent' results, approach so far still has too complicated a colour algebra.

Need to organize it better — introduce an extra 'small' parameter:

- **EITHER:** Assume that their angles are also strongly ordered.

  e.g. for $\omega_a \gg \omega_b$: $\theta_{a1} \gg \theta_{b1}$, or $\theta_{a1} \ll \theta_{b1}$

  Then exploit *coherence* to simplify colour.

- **OR:** Assume that $1/N_c^2$ is small *(large N_c limit)*. Then drop all terms with $1/N_c^2$ suppression.

Will talk about this only if there is time

Organizing squared amplitude

We have already assumed coupling is small and that gluon energies are *strongly ordered* ($\omega_b \ll \omega_a$).

To proceed towards simple all-orders 'incoherent' results, approach so far still has too complicated a colour algebra.

Need to organize it better — introduce an extra 'small' parameter:

- **EITHER:** Assume that their angles are also strongly ordered.
  e.g. for $\omega_a \gg \omega_b$: $\theta_{a1} \gg \theta_{b1}$, or $\theta_{a1} \ll \theta_{b1}$
  Then exploit *coherence* to simplify colour.
- **OR:** Assume that $1/N_c^2$ is small *(large N_c limit)*. Then drop all terms with $1/N_c^2$ suppression.

Will talk about this only if there is time

Organizing squared amplitude

We have already assumed coupling is small and that gluon energies are *strongly ordered* ($\omega_b \ll \omega_a$).

To proceed towards simple all-orders 'incoherent' results, approach so far still has too complicated a colour algebra.

Need to organize it better — introduce an extra 'small' parameter:

- **EITHER:** Assume that their angles are also strongly ordered.

  e.g. for $\omega_a \gg \omega_b$: $\theta_{a1} \gg \theta_{b1}$, or $\theta_{a1} \ll \theta_{b1}$

  Then exploit *coherence* to simplify colour.

- **OR:** Assume that $1/N_c^2$ is small *(large N_c limit)*. Then drop all terms with $1/N_c^2$ suppression.

  Will talk about this only if there is time

Organizing squared amplitude

We have already assumed coupling is small and that gluon energies are *strongly ordered* ($\omega_b \ll \omega_a$).

To proceed towards simple all-orders 'incoherent' results, approach so far still has too complicated a colour algebra.

Need to organize it better — introduce an extra 'small' parameter:

- **EITHER:** Assume that their angles are also strongly ordered.

  e.g. for $\omega_a \gg \omega_b$: $\theta_{a1} \gg \theta_{b1}$, or $\theta_{a1} \ll \theta_{b1}$

  Then exploit *coherence* to simplify colour.

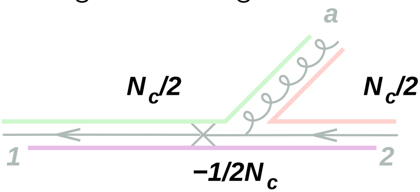
- **OR:** Assume that $1/N_c^2$ is small *(large N_c limit)*. Then drop all terms with $1/N_c^2$ suppression.

  Will talk about this only if there is time

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$

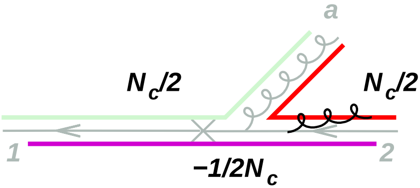


$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2}\left(C_F \frac{\Theta(\theta_{a2}-\theta_{b2})}{\theta_{b2}^2} + \right.$$
$$\left. + C_A \frac{\Theta(\theta_{a2}-\theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2}-\theta_{a2})}{\theta_{b2}^2}\right)$$

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$

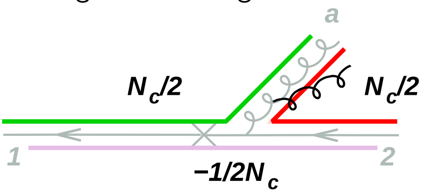


$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2}\left(C_F \frac{\Theta(\theta_{a2}-\theta_{b2})}{\theta_{b2}^2} + \right. \left. + C_A \frac{\Theta(\theta_{a2}-\theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2}-\theta_{a2})}{\theta_{b2}^2}\right)$$

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$



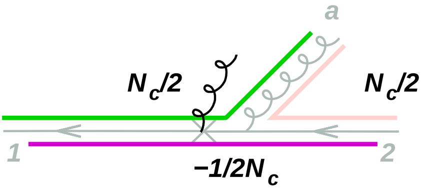
$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2} \left(C_F \frac{\Theta(\theta_{a2} - \theta_{b2})}{\theta_{b2}^2} + \right.$$
$$\left. + C_A \frac{\Theta(\theta_{a2} - \theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2} - \theta_{a2})}{\theta_{b2}^2} \right)$$

- Gluon b at large angle wrt $a2$ *does not resolve a2 structure.*
- It only sees overall colour charge of gluon+quark.
-

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$



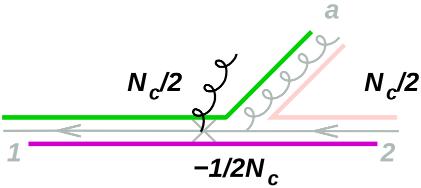
$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2}\left(C_F \frac{\Theta(\theta_{a2}-\theta_{b2})}{\theta_{b2}^2} + \right.$$
$$\left. + \, C_A \frac{\Theta(\theta_{a2}-\theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2}-\theta_{a2})}{\theta_{b2}^2}\right)$$

- Gluon b at large angle wrt $a2$ *does not resolve a2 structure.*
- It only sees overall colour charge of gluon+quark.
- By colour conservation this is just quark charge.

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$



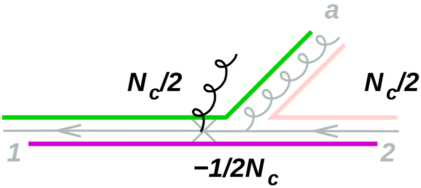
$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2}\left(C_F \frac{\Theta(\theta_{a2}-\theta_{b2})}{\theta_{b2}^2} + \right.$$
$$\left. + C_A \frac{\Theta(\theta_{a2}-\theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2}-\theta_{a2})}{\theta_{b2}^2}\right)$$

- Gluon *b* at large angle wrt *a*2 *does not resolve a2 structure.*
- It only sees overall colour charge of gluon+quark.
- By colour conservation this is just quark charge

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$



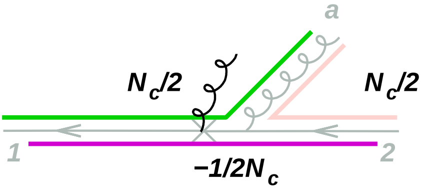
$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2}\left(C_F \frac{\Theta(\theta_{a2}-\theta_{b2})}{\theta_{b2}^2} + \right.$$
$$\left. + C_A \frac{\Theta(\theta_{a2}-\theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2}-\theta_{a2})}{\theta_{b2}^2}\right)$$

- Gluon b at large angle wrt $a2$ *does not resolve $a2$ structure*.
- It only sees overall colour charge of gluon+quark.
- By colour conservation this is just quark charge — COHERENCE

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$



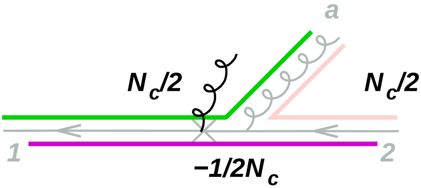
$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2}\left(C_F \frac{\Theta(\theta_{a2}-\theta_{b2})}{\theta_{b2}^2} + \right.$$
$$\left. + C_A \frac{\Theta(\theta_{a2}-\theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2}-\theta_{a2})}{\theta_{b2}^2}\right)$$

- Gluon b at large angle wrt $a2$ *does not resolve a2 structure*.
- It only sees overall colour charge of gluon+quark.
- By colour conservation this is just quark charge — COHERENCE

Angular ordering

Take all angles small. Require strong ordering between angles.

- $\theta_{b2} \ll \theta_{a2}$
- $\theta_{ba} \ll \theta_{a2}$
- $\theta_{ba} \sim \theta_{b2} \gg \theta_{a2}$



$$|M_{q\bar{q}ab}|^2 = |M_{q\bar{q}a}|^2 \frac{4g_s^2}{\omega_b^2} \left(C_F \frac{\Theta(\theta_{a2} - \theta_{b2})}{\theta_{b2}^2} + \right. \\ \left. + C_A \frac{\Theta(\theta_{a2} - \theta_{ab})}{\theta_{ab}^2} + C_F \frac{\Theta(\theta_{b2} - \theta_{a2})}{\theta_{b2}^2} \right)$$

- Gluon b at large angle wrt $a2$ *does not resolve a2 structure*.
- It only sees overall colour charge of gluon+quark.
- By colour conservation this is just quark charge — COHERENCE

Angular ordering @ all orders

Coherence:

- large-angle radiation doesn't resolve what happens at small angles.
- small-angle radiation doesn't care what is going on at larger angles.
- gluons on widely disparate angular scales do not 'talk to each other.'

Matrix element for n-gluon emission (strong θ and ω ordering)

$$|M_{q\bar{q}\,g_1\ldots g_n}|^2 \simeq |M_{q\bar{q}}|^2 \prod_{i=1}^{n} \frac{4g_s^2}{\omega_i^2} \frac{C_S\,\Theta(\omega_{i-1}-\omega_i)}{\min(\theta_{qg_i},\theta_{\bar{q}g_i},\theta_{g_1g_i},\ldots,\theta_{g_{i-1}g_i})^2}$$

This is simplified version of a classic QCD result.

[Can be cast in many ways]

C_S: find the 'harder' (q, $\bar{q}$ or $j<i$) parton j that gives the minimum angle in the denominator. Find the set S consisting of j and of all other partons k (q, $\bar{q}$ or $k<i$), that satisfy $\theta_{kj}<\theta_{ij}$. C_S is the overall colour charge (C_F, C_A) of that set.

Angular ordering @ all orders

Coherence:

- large-angle radiation doesn't resolve what happens at small angles.
- small-angle radiation doesn't care what is going on at larger angles.
- gluons on widely disparate angular scales do not 'talk to each other.'

Matrix element for n-gluon emission (strong θ and ω ordering)

$$|M_{q\bar{q}\,g_1\ldots g_n}|^2 \simeq |M_{q\bar{q}}|^2 \prod_{i=1}^{n} \frac{4g_s^2}{\omega_i^2} \frac{C_S\,\Theta(\omega_{i-1}-\omega_i)}{\min(\theta_{qg_i},\theta_{\bar{q}g_i},\theta_{g_1g_i},\ldots,\theta_{g_{i-1}g_i})^2}$$

This is simplified version of a classic QCD result.

[Can be cast in many ways]

C_S: find the 'harder' (q, $\bar{q}$ or $j < i$) parton j that gives the minimum angle in the denominator. Find the set S consisting of j and of all other partons k (q, $\bar{q}$ or $k < i$), that satisfy $\theta_{kj} < \theta_{ij}$. C_S is the overall colour charge (C_F, C_A) of that set.

Angular ordering @ all orders

Coherence:

- large-angle radiation doesn't resolve what happens at small angles.
- small-angle radiation doesn't care what is going on at larger angles.
- gluons on widely disparate angular scales do not 'talk to each other.'

Matrix element for n-gluon emission (strong θ and ω ordering)

$$|M_{q\bar{q}\,g_1\ldots g_n}|^2 \simeq |M_{q\bar{q}}|^2 \prod_{i=1}^{n} \frac{4g_s^2}{\omega_i^2} \frac{C_S\,\Theta(\omega_{i-1}-\omega_i)}{\min(\theta_{qg_i}, \theta_{\bar{q}g_i}, \theta_{g_1 g_i}, \ldots, \theta_{g_{i-1}g_i})^2}$$

This is simplified version of a classic QCD result.

[Can be cast in many ways]

C_S: find the 'harder' (q, $\bar{q}$ or $j < i$) parton j that gives the minimum angle in the denominator. Find the set S consisting of j and of all other partons k (q, $\bar{q}$ or $k < i$), that satisfy $\theta_{kj} < \theta_{ij}$. C_S is the overall colour charge (C_F, C_A) of that set.

Angular ordering as a cascade

Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_F \alpha_s \frac{d\theta}{\theta} \frac{d\omega}{\omega}$ structure) *radiate* a gluon

Angular ordering as a cascade

Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_S \alpha_s \frac{d\theta}{\theta} \frac{d\omega}{\omega}$ structure) *radiate* a gluon.
4. Go to step 2, but with *two independent emitters* — the original one and the newly radiated gluon.
5. *Stop* on a given emitter when you reach the smallest perturbatively allowed angle $\theta \sim \Lambda / E_{emitter}$.

Highly successful Monte Carlo 'event generators' (PYTHIA, HERWIG, etc.) use such principles.

Angular ordering as a cascade

Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_S \alpha_s \frac{d\theta}{\theta} \frac{d\omega}{\omega}$ structure) *radiate* a gluon.
4. Go to step 2, but with *two independent emitters* — the original one and the newly radiated gluon.
5. *Stop* on a given emitter when you reach the smallest perturbatively allowed angle $\theta \sim \Lambda / E_{emitter}$.

Highly successful Monte Carlo 'event generators' (PYTHIA, HERWIG, etc.) use such principles.

Angular ordering as a cascade

Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_S \alpha_s \frac{d\theta}{\theta} \frac{d\omega}{\omega}$ structure) *radiate* a gluon.
4. Go to step 2, but with *two independent emitters* — the original one and the newly radiated gluon.
5. *Stop* on a given emitter when you reach the smallest perturbatively allowed angle $\theta \sim \Lambda / E_{emitter}$.

Highly successful Monte Carlo 'event generators' (PYTHIA, HERWIG, etc.) use such principles.

Angular ordering as a cascade

Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_S \alpha_s \frac{d\theta}{\theta}\frac{d\omega}{\omega}$ structure) *radiate* a gluon.
4. Go to step 2, but with *two independent emitters* — the original one and the newly radiated gluon.
5. *Stop* on a given emitter when you reach the smallest perturbatively allowed angle $\theta \sim \Lambda/E_{emitter}$.

Highly successful Monte Carlo 'event generators' (PYTHIA, HERWIG, etc.) use such principles.

Angular ordering as a cascade

Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_S \alpha_s \frac{d\theta}{\theta} \frac{d\omega}{\omega}$ structure) *radiate* a gluon.
4. Go to step 2, but with *two independent emitters* — the original one and the newly radiated gluon.
5. *Stop* on a given emitter when you reach the smallest perturbatively allowed angle $\theta \sim \Lambda / E_{emitter}$.

Highly successful Monte Carlo 'event generators' (PYTHIA, HERWIG, etc.) use such principles.

Angular ordering as a cascade

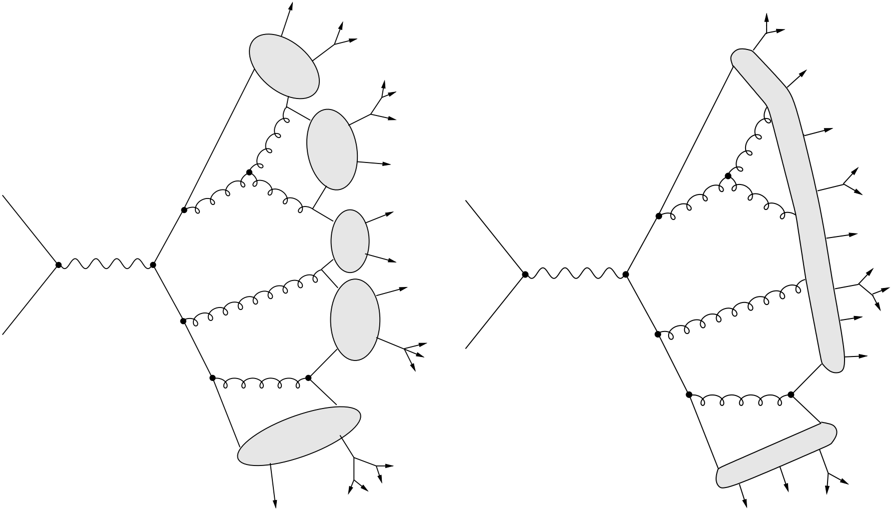
Angular ordering can be expressed in many ways. Most powerful (perhaps) is in form of a cascade.

This is basis for *simulations* of QCD multi-gluon emission.

1. Start with a quark as the *'emitter'*.
2. *'Scan'* towards small angles.
3. At 'some point' (based on $C_S \alpha_s \frac{d\theta}{\theta} \frac{d\omega}{\omega}$ structure) *radiate* a gluon.
4. Go to step 2, but with *two independent emitters* — the original one and the newly radiated gluon.
5. *Stop* on a given emitter when you reach the smallest perturbatively allowed angle $\theta \sim \Lambda / E_{emitter}$.

Highly successful Monte Carlo 'event generators' (PYTHIA, HERWIG, etc.) use such principles.

Hadronisation



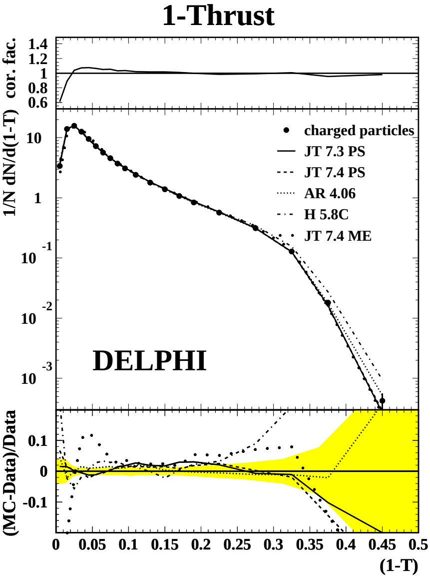
[fig. from B.R. Webber]

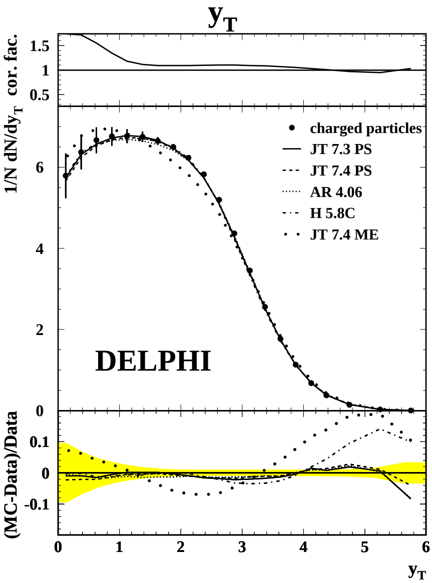
3rd 'problem' (not on problem sheet)

Lattice QCD can today calculate non-perturbative effects for the Υ system (a $b\bar{b}$ bound state), where the ratio of hard to soft scales is about 10 (the typical momentum scales are not m_b but $\alpha_s m_b$).

Make a guess as to how the difficulty of lattice calculations depends on the ratio of hard to soft scales. Assuming the continued validity of Moore's law (computing power doubles every 18 months), how long will it be before lattice can give a direct calculation of hadronisation effects in high-energy (100 GeV) collisions?

'Monte Carlo' comparisons





Why strongly ordered angles and energies?

It's easier, but is it justified?!

Answer: it depends on what you look at.

Use 'bootstrap' arguments. Calculate with 1-gluon soft-collinear approx:

✓ = dominated by soft-collinear region
✗ = dominated by hard region (→ NLO)

Results

✗ number of events with 3 hard jets

$$\sigma_{3-jet} \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\Theta\left(\frac{\omega}{Q}-\epsilon\right)\Theta(\theta-\delta)$$

✗ *mean* value of 1−Thrust

$$\langle 1-T\rangle \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\frac{\omega\theta^2}{2Q}$$

Why strongly ordered angles and energies?

It's easier, but is it justified?!

Answer: it depends on what you look at.

Use 'bootstrap' arguments. Calculate with 1-gluon soft-collinear approx:

✓ = dominated by soft-collinear region

✗ = dominated by hard region (→ NLO)

Results

✗ number of events with 3 hard jets

$$\sigma_{3-jet} \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\Theta\left(\frac{\omega}{Q} - \epsilon\right)\Theta(\theta - \delta)$$

✗ *mean* value of 1−Thrust

$$\langle 1 - T \rangle \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\frac{\omega\theta^2}{2Q}$$

number of 'subjets' inside a jet

typical value of 1−Thrust

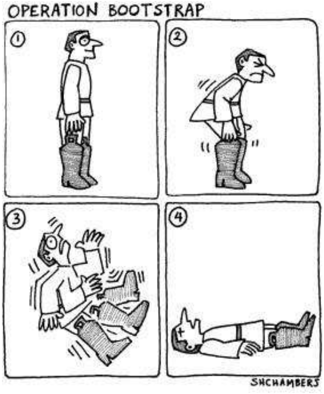
Why strongly ordered angles and energies?

It's easier, but is it justified?!

Answer: it depends on what you look at.

Use 'bootstrap' arguments. Calculate with 1-gluon soft-collinear approx:

✓ = dominated by soft-collinear region
✗ = dominated by hard region (→ NLO)



Results

✗ number of events with 3 hard jets

$$\sigma_{3-jet} \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\Theta\left(\frac{\omega}{Q} - \epsilon\right)\Theta(\theta-\delta)$$

✗ *mean* value of 1−Thrust

$$\langle 1 - T\rangle \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\frac{\omega\theta^2}{2Q}$$

✓ number of 'subjets' inside a jet

✓ *typical* value of 1−Thrust

Why strongly ordered angles and energies?

It's easier, but is it justified?!

Answer: it depends on what you look at.

Use 'bootstrap' arguments. Calculate with 1-gluon soft-collinear approx:

✓ = dominated by soft-collinear region
✗ = dominated by hard region (→ NLO)

Results

✗ number of events with 3 hard jets

$$\sigma_{3-jet} \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\Theta\left(\frac{\omega}{Q}-\epsilon\right)\Theta(\theta-\delta)$$

✗ *mean* value of 1−Thrust

$$\langle 1-T\rangle \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\frac{\omega\theta^2}{2Q}$$

✓ number of 'subjets' inside a jet

✓ *typical* value of 1−Thrust

Why strongly ordered angles and energies?

It's easier, but is it justified?!

Answer: it depends on what you look at.

Use 'bootstrap' arguments. Calculate with 1-gluon soft-collinear approx:

✓ = dominated by soft-collinear region
✗ = dominated by hard region (→ NLO)

Results

✗ number of events with 3 hard jets

$$\sigma_{3-jet} \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\Theta\left(\frac{\omega}{Q}-\epsilon\right)\Theta(\theta-\delta)$$

✗ *mean* value of 1−Thrust

$$\langle 1-T\rangle \sim \alpha_s \int \frac{d\omega}{\omega}\frac{d\theta}{\theta}\frac{\omega\theta^2}{2Q}$$

✓ number of 'subjets' inside a jet

✓ *typical* value of 1−Thrust

Thrust — a QCD ‘guinea pig’

First discussion goes back to 1964. Serious work got going in late ’70s.
Thrust is one of many continous measures of the *event ‘shape’:*

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i . \vec{n}_T|}{\sum_i |\vec{p}_i|} \,,$$

2-jet event: $T \simeq 1$

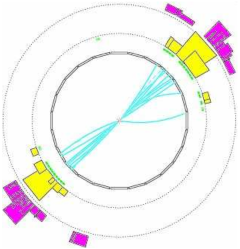
3-jet event: $T \simeq 2/3$

There exist many other measures of aspects of the shape: Thrust-Major, C-parameter, broadening, heavy-jet mass, jet-resolution parameters, . . .

Thrust — a QCD 'guinea pig'

First discussion goes back to 1964. Serious work got going in late '70s.
Thrust is one of many continous measures of the *event 'shape':*

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i.\vec{n}_T|}{\sum_i |\vec{p}_i|},$$



2-jet event: $T \simeq 1$

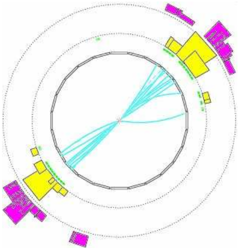
3-jet event: $T \simeq 2/3$

There exist many other measures of aspects of the shape: Thrust-Major, C-parameter, broadening, heavy-jet mass, jet-resolution parameters,...

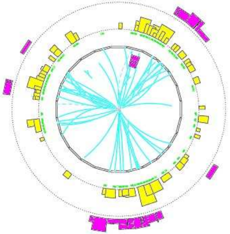
Thrust — a QCD 'guinea pig'

First discussion goes back to 1964. Serious work got going in late '70s. Thrust is one of many continous measures of the *event 'shape':*

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i.\vec{n}_T|}{\sum_i |\vec{p}_i|},$$



2-jet event: $T \simeq 1$



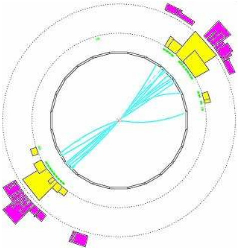
3-jet event: $T \simeq 2/3$

There exist many other measures of aspects of the shape: Thrust-Major, C-parameter, broadening, heavy-jet mass, jet-resolution parameters,...

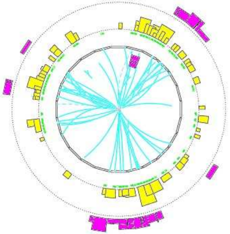
Thrust — a QCD ‘guinea pig’

First discussion goes back to 1964. Serious work got going in late ’70s.
Thrust is one of many continous measures of the *event ‘shape’:*

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i . \vec{n}_T|}{\sum_i |\vec{p}_i|},$$



2-jet event: $T \simeq 1$



3-jet event: $T \simeq 2/3$

There exist many other measures of aspects of the shape: Thrust-Major, C-parameter, broadening, heavy-jet mass, jet-resolution parameters,...

Thrust mean

Thrust in soft & collinear (SC) limit: actually discuss $1 - T$ since this measures *deviation* from $q\bar{q}$ case:

$$1 - T \simeq \sum_{i \in \text{soft}} \frac{\omega_i \theta_i^2}{2Q}$$

θ_i should be wrt thrust axis, but can replace it with angle wrt quark axis

Hence SC approx. for mean value is poor since dominated by large ω, θ:

$$\langle 1 - T \rangle \sim \frac{2\alpha_s C_F}{\pi} \int \frac{d\omega}{\omega} \frac{d\theta}{\theta} \cdot \frac{\omega\theta^2}{2Q}$$

Need full hard corrections both to matrix element and to thrust value.

Thrust mean

Thrust in soft & collinear (SC) limit: actually discuss $1 - T$ since this measures *deviation* from $q\bar{q}$ case:

$$1 - T \simeq \sum_{i \in \text{soft}} \frac{\omega_i \theta_i^2}{2Q}$$

θ_i should be wrt thrust axis, but can replace it with angle wrt quark axis

Hence SC approx. for mean value is poor since dominated by large ω, θ:

$$\langle 1 - T \rangle \sim \frac{2\alpha_s C_F}{\pi} \int \frac{d\omega}{\omega} \frac{d\theta}{\theta} \cdot \frac{\omega\theta^2}{2Q}$$

Need full hard corrections both to matrix element and to thrust value.

Thrust median (typical value)

Find value τ of $1-T$ such that $\sigma(1-T>\tau)=\frac{1}{2}\sigma_{tot}$. Solve

$$\frac{4\alpha_s C_F}{\pi}\int^{\pi/2}\frac{d\theta}{\theta}\int^{Q}\frac{d\omega}{\omega}\Theta\left(\frac{\omega\theta^2}{2Q}-\tau\right)=\frac{1}{2}$$

Two logarithmic integrals. To establish whether soft & collinear part dominates, consider solution for τ:

$$\frac{\alpha_s C_F}{\pi}\ln^2\tau+\mathcal{O}(\alpha_s\ln\tau)=\frac{1}{2}\quad\to\quad\ln\frac{1}{\tau}\simeq\sqrt{\frac{\pi}{\alpha_s C_F}}\quad\to\quad\tau\sim e^{-\sqrt{\frac{\pi}{\alpha_s C_F}}}$$

- $\tau \ll 1$, so soft-collinear approximation was *valid* ('bootstrap' OK)
- α_s still small despite $\tau \ll 1$, since $\frac{\alpha_s}{1-\alpha_s b\ln\tau}\sim\frac{\alpha_s}{1-b\sqrt{\pi\alpha_s/C_F}}$
-

Thrust median (typical value)

Find value τ of $1-T$ such that $\sigma(1-T>\tau) = \frac{1}{2}\sigma_{tot}$. Solve

$$\frac{4\alpha_s C_F}{\pi}\int^{\pi/2}\frac{d\theta}{\theta}\int^{Q}\frac{d\omega}{\omega}\Theta\left(\frac{\omega\theta^2}{2Q}-\tau\right) = \frac{1}{2}$$

Two logarithmic integrals. To establish whether soft & collinear part dominates, consider solution for τ:

$$\frac{\alpha_s C_F}{\pi}\ln^2\tau + \mathcal{O}(\alpha_s \ln\tau) = \frac{1}{2} \quad\to\quad \ln\frac{1}{\tau} \simeq \sqrt{\frac{\pi}{\alpha_s C_F}} \quad\to\quad \tau \sim e^{-\sqrt{\frac{\pi}{\alpha_s C_F}}}$$

- $\tau \ll 1$, so soft-collinear approximation was *valid* ('bootstrap' OK)
- α_s still small despite $\tau \ll 1$, since $\frac{\alpha_s}{1-\alpha_s b \ln\tau} \sim \frac{\alpha_s}{1-b\sqrt{\pi\alpha_s/C_F}}$
- *Double logarithmic structure* ($\alpha_s \ln^2\tau$) *is typical of QCD (and QED)*

Thrust median (typical value)

Find value τ of $1 - T$ such that $\sigma(1 - T > \tau) = \frac{1}{2}\sigma_{tot}$. Solve

$$\frac{4\alpha_s C_F}{\pi} \int^{\pi/2} \frac{d\theta}{\theta} \int^{Q} \frac{d\omega}{\omega} \Theta\left(\frac{\omega\theta^2}{2Q} - \tau\right) = \frac{1}{2}$$

Two logarithmic integrals. To establish whether soft & collinear part dominates, consider solution for τ:

$$\frac{\alpha_s C_F}{\pi} \ln^2 \tau + \mathcal{O}(\alpha_s \ln \tau) = \frac{1}{2} \quad \rightarrow \quad \ln \frac{1}{\tau} \simeq \sqrt{\frac{\pi}{\alpha_s C_F}} \quad \rightarrow \quad \tau \sim e^{-\sqrt{\frac{\pi}{\alpha_s C_F}}}$$

- $\tau \ll 1$, so soft-collinear approximation was *valid* ('bootstrap' OK)
- α_s still small despite $\tau \ll 1$, since $\frac{\alpha_s}{1-\alpha_s b \ln \tau} \sim \frac{\alpha_s}{1-b\sqrt{\pi\alpha_s/C_F}}$
- *Double logarithmic structure* ($\alpha_s \ln^2 \tau$) is typical of QCD (and QED)

Thrust median (typical value)

Find value τ of $1-T$ such that $\sigma(1-T>\tau)=\frac{1}{2}\sigma_{tot}$. Solve

$$\frac{4\alpha_s C_F}{\pi}\int^{\pi/2}\frac{d\theta}{\theta}\int^{Q}\frac{d\omega}{\omega}\Theta\left(\frac{\omega\theta^2}{2Q}-\tau\right)=\frac{1}{2}$$

Two logarithmic integrals. To establish whether soft & collinear part dominates, consider solution for τ:

$$\frac{\alpha_s C_F}{\pi}\ln^2\tau+\mathcal{O}(\alpha_s\ln\tau)=\frac{1}{2}\quad\rightarrow\quad\ln\frac{1}{\tau}\simeq\sqrt{\frac{\pi}{\alpha_s C_F}}\quad\rightarrow\quad\tau\sim e^{-\sqrt{\frac{\pi}{\alpha_s C_F}}}$$

- $\tau \ll 1$, so soft-collinear approximation was *valid* ('bootstrap' OK)
- α_s still small despite $\tau\ll 1$, since $\frac{\alpha_s}{1-\alpha_s b\ln\tau}\sim\frac{\alpha_s}{1-b\sqrt{\pi\alpha_s/C_F}}$
- *Double logarithmic structure* ($\alpha_s\ln^2\tau$) is typical of QCD (and QED)

Thrust median (typical value)

Find value τ of $1-T$ such that $\sigma(1-T>\tau)=\frac{1}{2}\sigma_{tot}$. Solve

$$\frac{4\alpha_s C_F}{\pi}\int^{\pi/2}\frac{d\theta}{\theta}\int^{Q}\frac{d\omega}{\omega}\Theta\left(\frac{\omega\theta^2}{2Q}-\tau\right)=\frac{1}{2}$$

Two logarithmic integrals. To establish whether soft & collinear part dominates, consider solution for τ:

$$\frac{\alpha_s C_F}{\pi}\ln^2\tau+\mathcal{O}\left(\alpha_s\ln\tau\right)=\frac{1}{2}\quad\rightarrow\quad\ln\frac{1}{\tau}\simeq\sqrt{\frac{\pi}{\alpha_s C_F}}\quad\rightarrow\quad\tau\sim e^{-\sqrt{\frac{\pi}{\alpha_s C_F}}}$$

- $\tau\ll 1$, so soft-collinear approximation was *valid* ('bootstrap' OK)
- α_s still small despite $\tau\ll 1$, since $\frac{\alpha_s}{1-\alpha_s b\ln\tau}\sim\frac{\alpha_s}{1-b\sqrt{\pi\alpha_s/C_F}}$
- *Double logarithmic structure* ($\alpha_s\ln^2\tau$) is typical of QCD (and QED)

'Typical' versus 'hard' events

Results like those of previous pages are a regular occurrence:

- weighting with 'hardness' (*e.g.* $\langle 1 - T \rangle$) selects events with hard radiation: *rare* — suppressed by power of α_s
- typical events, by definition, not suppressed by α_s.

  ➥ α_s from radiation is compensated by $\log^2$ from divergent soft-collinear phase space
- soft-collinear $\log^2$ accompanies *each* power of α_s:

  $$\text{Perturbative series} \sim \sum_n (\alpha_s \ln^2 \tau)^n$$

  Since $\ln^2 \tau \sim 1/\alpha_s$, this series **must not be truncated**.
- Understanding of all-order properties of divergent soft & collinear branching not only 'interesting', but essential.

'Typical' versus 'hard' events

Results like those of previous pages are a regular occurrence:

- weighting with 'hardness' (*e.g.* $\langle 1 - T \rangle$) selects events with hard radiation: *rare* — suppressed by power of α_s
- typical events, by definition, not suppressed by α_s.

➥ α_s from radiation is compensated by $\log^2$ from divergent soft-collinear phase space

- soft-collinear $\log^2$ accompanies *each* power of α_s:

$$\text{Perturbative series} \sim \sum_n (\alpha_s \ln^2 \tau)^n$$

Since $\ln^2 \tau \sim 1/\alpha_s$, this series **must not be truncated**.

- Understanding of all-order properties of divergent soft & collinear branching not only 'interesting', but essential.

‘Typical’ versus ‘hard’ events

Results like those of previous pages are a regular occurrence:

- weighting with ‘hardness’ (*e.g.* $\langle 1 - T \rangle$) selects events with hard radiation: *rare* — suppressed by power of α_s
- typical events, by definition, not suppressed by α_s.

  ➥ α_s from radiation is compensated by $\log^2$ from divergent soft-collinear phase space

- soft-collinear $\log^2$ accompanies *each* power of α_s:

  Perturbative series $\sim \sum_n (\alpha_s \ln^2 \tau)^n$

  Since $\ln^2 \tau \sim 1/\alpha_s$, this series **must not be truncated**.

- Understanding of all-order properties of divergent soft & collinear branching not only ‘interesting’, but essential.

‘Typical’ versus ‘hard’ events

Results like those of previous pages are a regular occurrence:

- weighting with ‘hardness’ (*e.g.* $\langle 1-T \rangle$) selects events with hard radiation: *rare* — suppressed by power of α_s
- typical events, by definition, not suppressed by α_s.

  ➥ α_s from radiation is compensated by $\log^2$ from divergent soft-collinear phase space

- soft-collinear $\log^2$ accompanies *each* power of α_s:

  Perturbative series $\sim \sum_n (\alpha_s \ln^2 \tau)^n$

  Since $\ln^2 \tau \sim 1/\alpha_s$, this series **must not be truncated**.

- Understanding of all-order properties of divergent soft & collinear branching not only ‘interesting’, but essential.

‘Typical’ versus ‘hard’ events

Results like those of previous pages are a regular occurrence:

- weighting with ‘hardness’ (*e.g.* $\langle 1 - T \rangle$) selects events with hard radiation: *rare* — suppressed by power of α_s
- typical events, by definition, not suppressed by α_s.

➥ α_s from radiation is compensated by $\log^2$ from divergent soft-collinear phase space

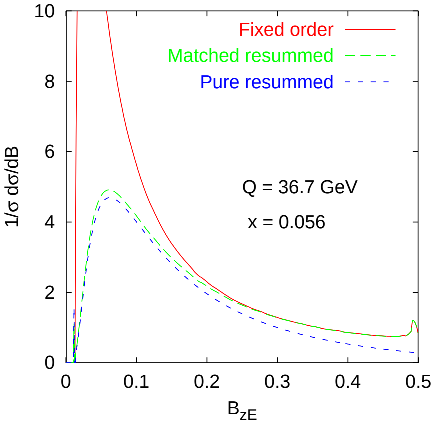
- soft-collinear $\log^2$ accompanies *each* power of α_s:

  Perturbative series $\sim \sum_n (\alpha_s \ln^2 \tau)^n$

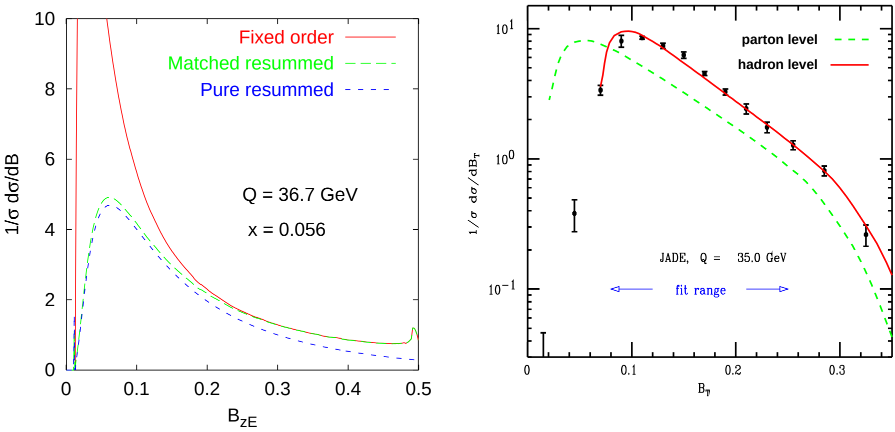
  Since $\ln^2 \tau \sim 1/\alpha_s$, this series **must not be truncated**.
- Understanding of all-order properties of divergent soft & collinear branching not only ‘interesting’, but essential.

This is called *resummation*

Resummation



Resummation



Sort of a summary

Range of QCD tools:

- Fixed-order calculations (LO, NLO, . . .)
- Calculations based on multiple soft-collinear radiation
  - Monte Carlo 'cascade' event generators
  - analytical resummations $\sum(\alpha_s \ln^2 \tau)^n$
- Well-defined non-perturbative inputs (structure functions, fragmentation functions).
- Ill-defined non-perturbative inputs (hadronisation)
  - Monte Carlo hadronisation models (many parameters, good fits)
  - Analytical models (1 parameter, sometimes good fits!)

Sort of a summary

Range of QCD tools:

- Fixed-order calculations (LO, NLO, ...)
- Calculations based on multiple soft-collinear radiation
  - Monte Carlo 'cascade' event generators
  - analytical resummations $\sum(\alpha_s \ln^2 \tau)^n$
- Well-defined non-perturbative inputs (structure functions, fragmentation functions).
- Ill-defined non-perturbative inputs (hadronisation)
  - Monte Carlo hadronisation models (many parameters, good fits)
  - Analytical models (1 parameter, sometimes good fits!)

Sort of a summary

Range of QCD tools:

- Fixed-order calculations (LO, NLO, . . .)
- Calculations based on multiple soft-collinear radiation
  - Monte Carlo 'cascade' event generators
  - analytical resummations $\sum(\alpha_s \ln^2 \tau)^n$
- Well-defined non-perturbative inputs (structure functions, fragmentation functions).
- Ill-defined non-perturbative inputs (hadronisation)
  - Monte Carlo hadronisation models (many parameters, good fits)
  - Analytical models (1 parameter, sometimes good fits!)

Sort of a summary

Range of QCD tools:

- Fixed-order calculations (LO, NLO, . . .)
- Calculations based on multiple soft-collinear radiation
  - Monte Carlo 'cascade' event generators
  - analytical resummations $\sum(\alpha_s \ln^2 \tau)^n$
- Well-defined non-perturbative inputs (structure functions, fragmentation functions).
- Ill-defined non-perturbative inputs (hadronisation)
  - Monte Carlo hadronisation models (many parameters, good fits)
  - Analytical models (1 parameter, sometimes good fits!)

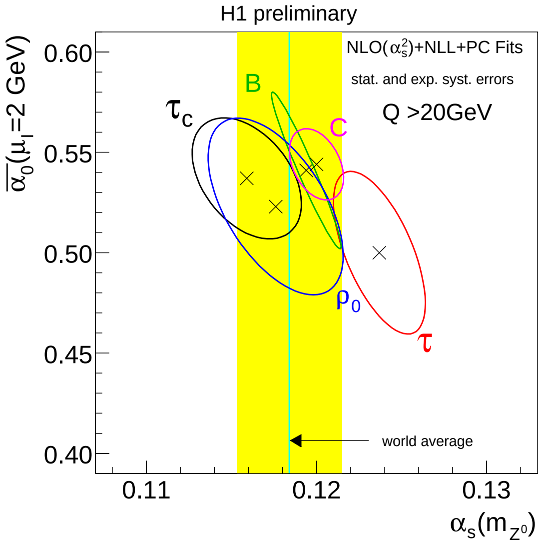
Sort of a summary

Range of QCD tools:

- Fixed-order calculations (LO, NLO, ...)
- Calculations based on multiple soft-collinear radiation
  - Monte Carlo 'cascade' event generators
  - analytical resummations $\sum(\alpha_s \ln^2 \tau)^n$
- Well-defined non-perturbative inputs (structure functions, fragmentation functions).
- Ill-defined non-perturbative inputs (hadronisation)
  - Monte Carlo hadronisation models (many parameters, good fits)
  - Analytical models (1 parameter, sometimes good fits!)

EXTRA SLIDES

Fitting hadronisation with a single parameter



Jet clustering algorithms (for reference)

1. Define a distance measure for every pair of (pseudo)particles i and j

$$y_{ij} = \min(E_i^2, E_j^2)\,(1 - \cos\theta) \qquad \text{'Durham' or '}k_t\text{' measure}$$

2. Find the pair of (pseudo)particles with the smallest y_{ij}. If this y_{ij} is larger than some threshold y_{cut}, then stop.

3. Otherwise recombine i and j into a single 'pseudo-particle' and go to step 1.