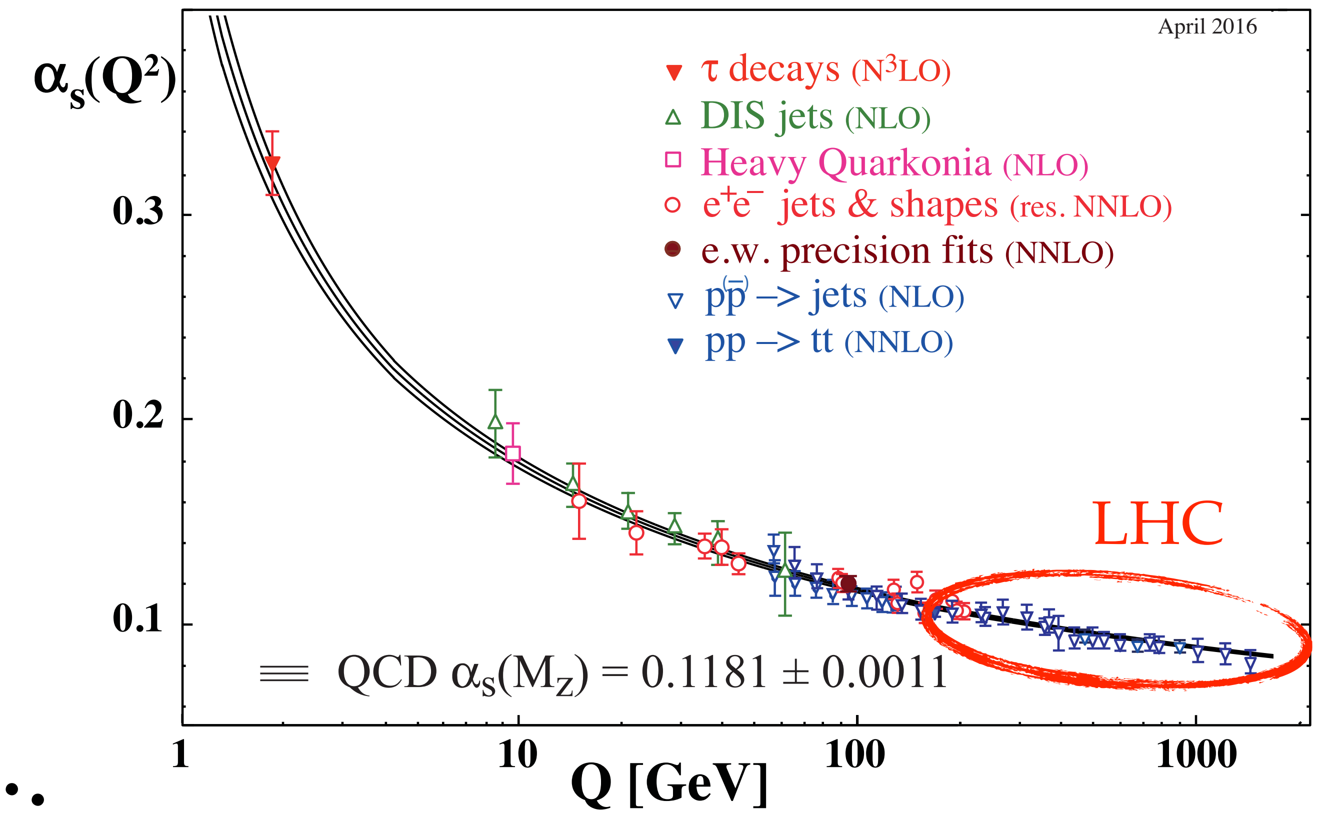
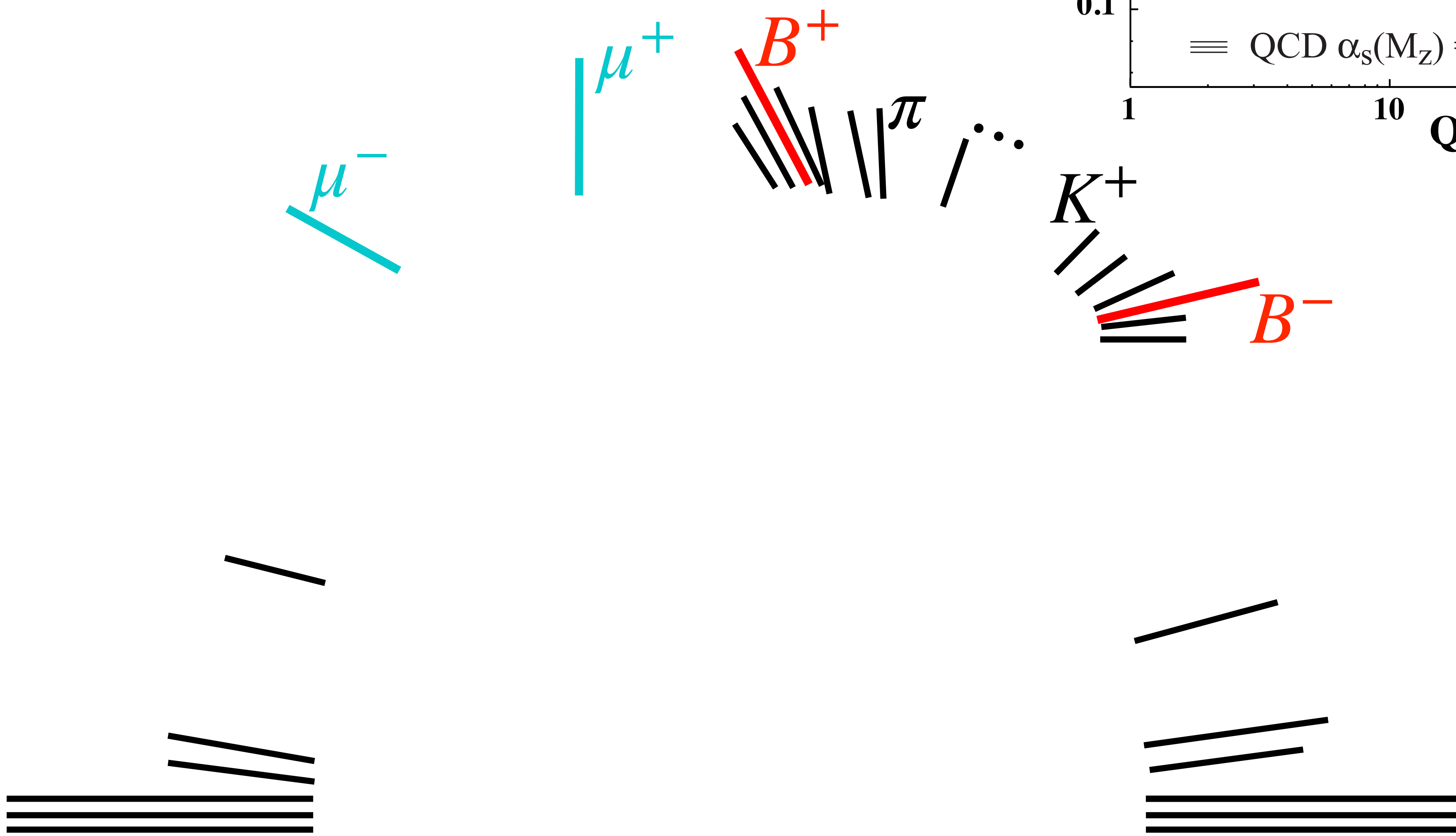
Introduction to QCD at high-energy colliders
Lectures 4 & 5: jets

Gavin Salam, Oxford, February 2024
as part of the QCD PhD course with
Fabrizio Caola, Jack Helliwell and Peter Skands

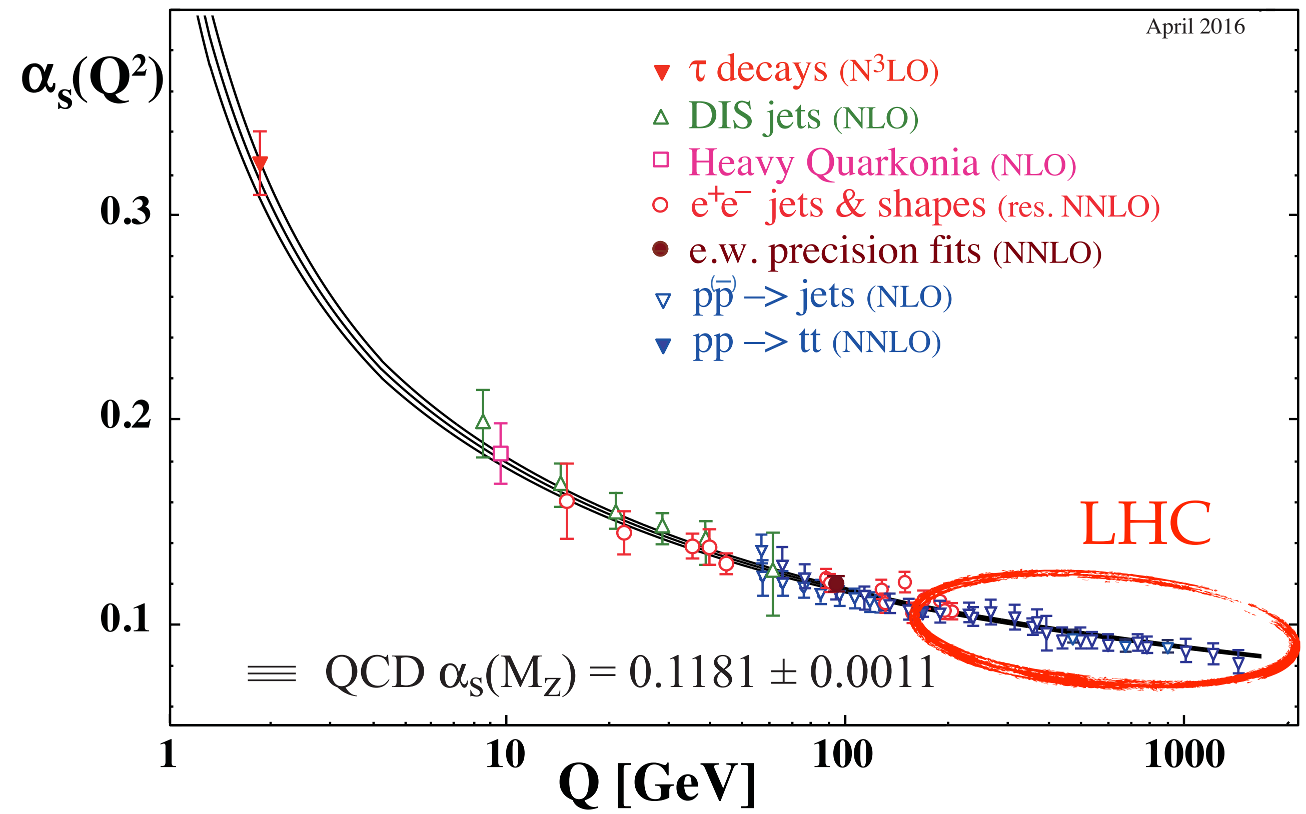
(with extensive use of material by
Matteo Cacciari and Gregory Soyez)

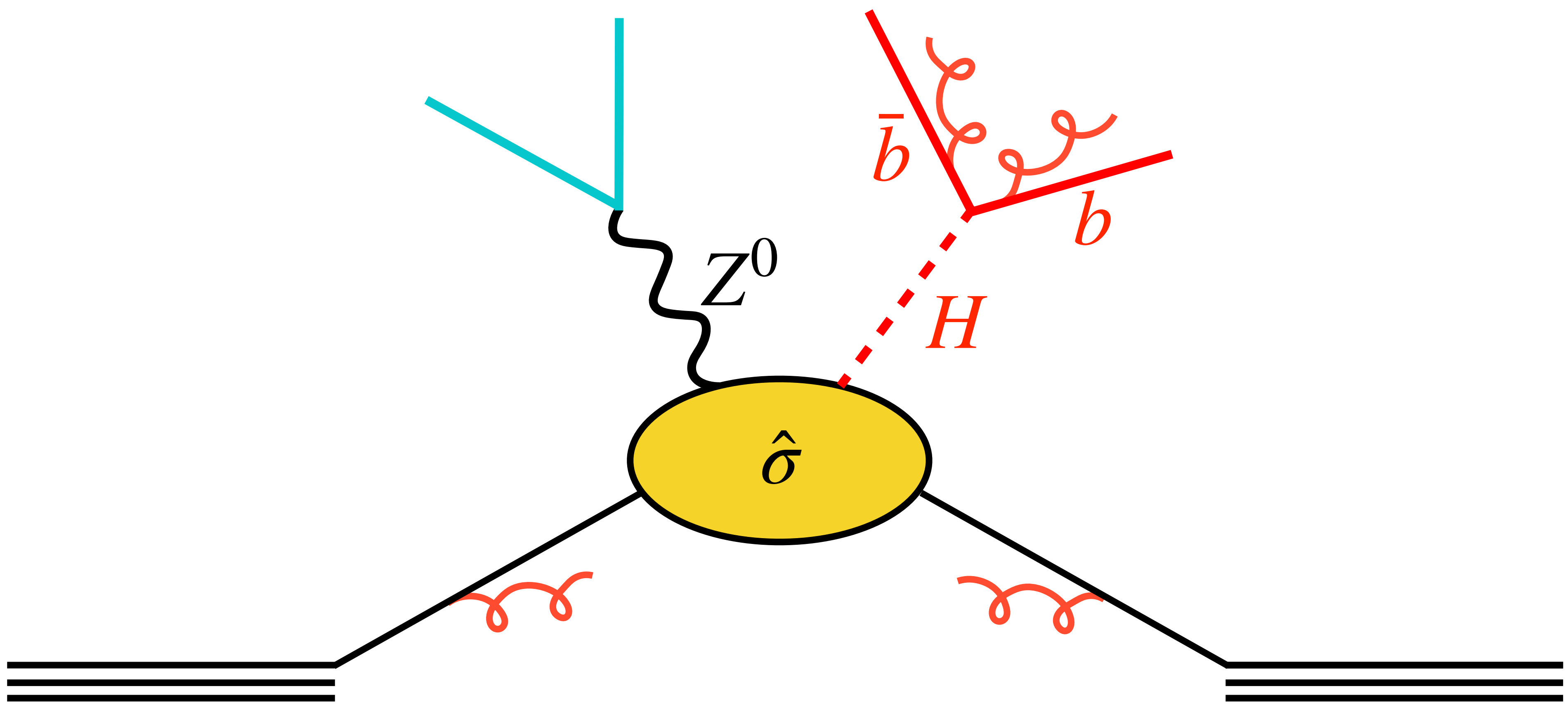
strong coupling v. Q





strong coupling v. Q





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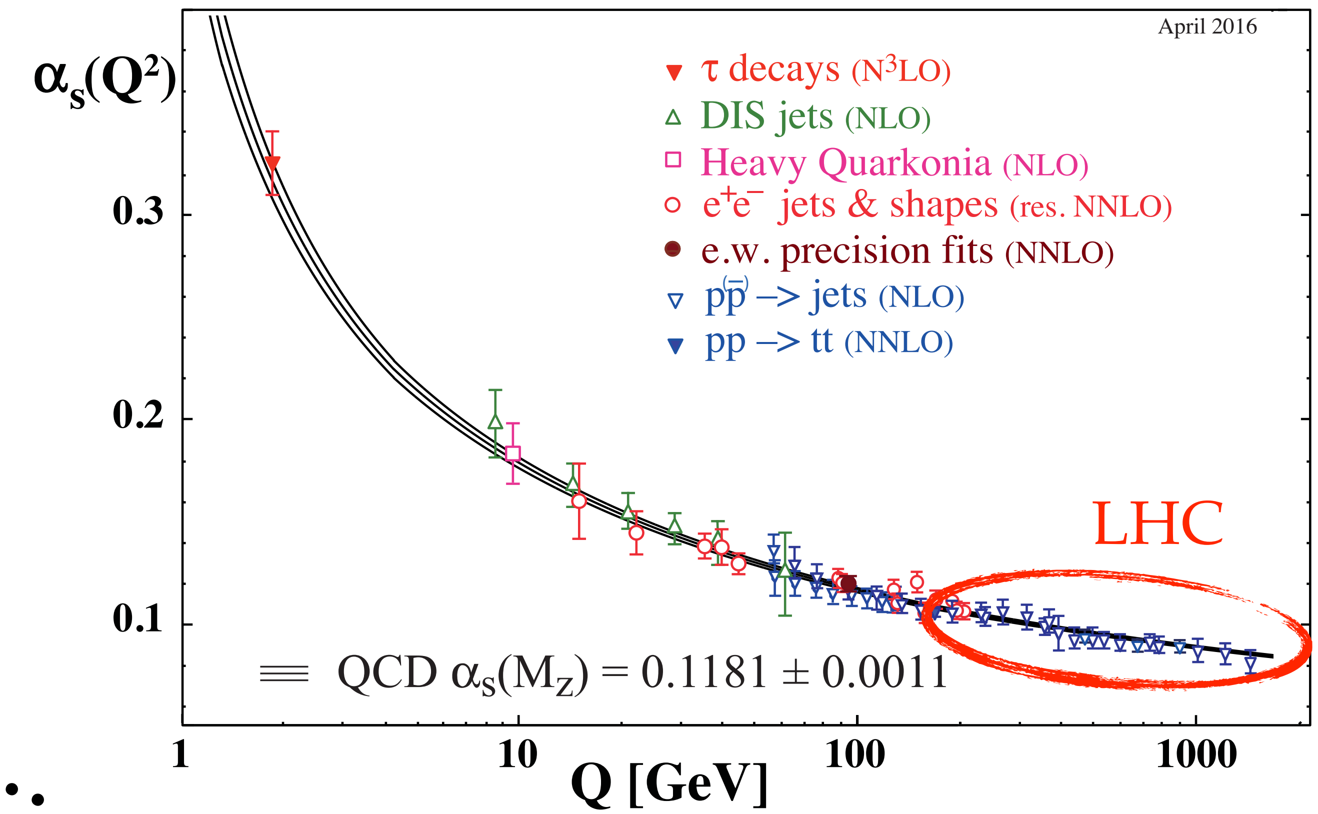
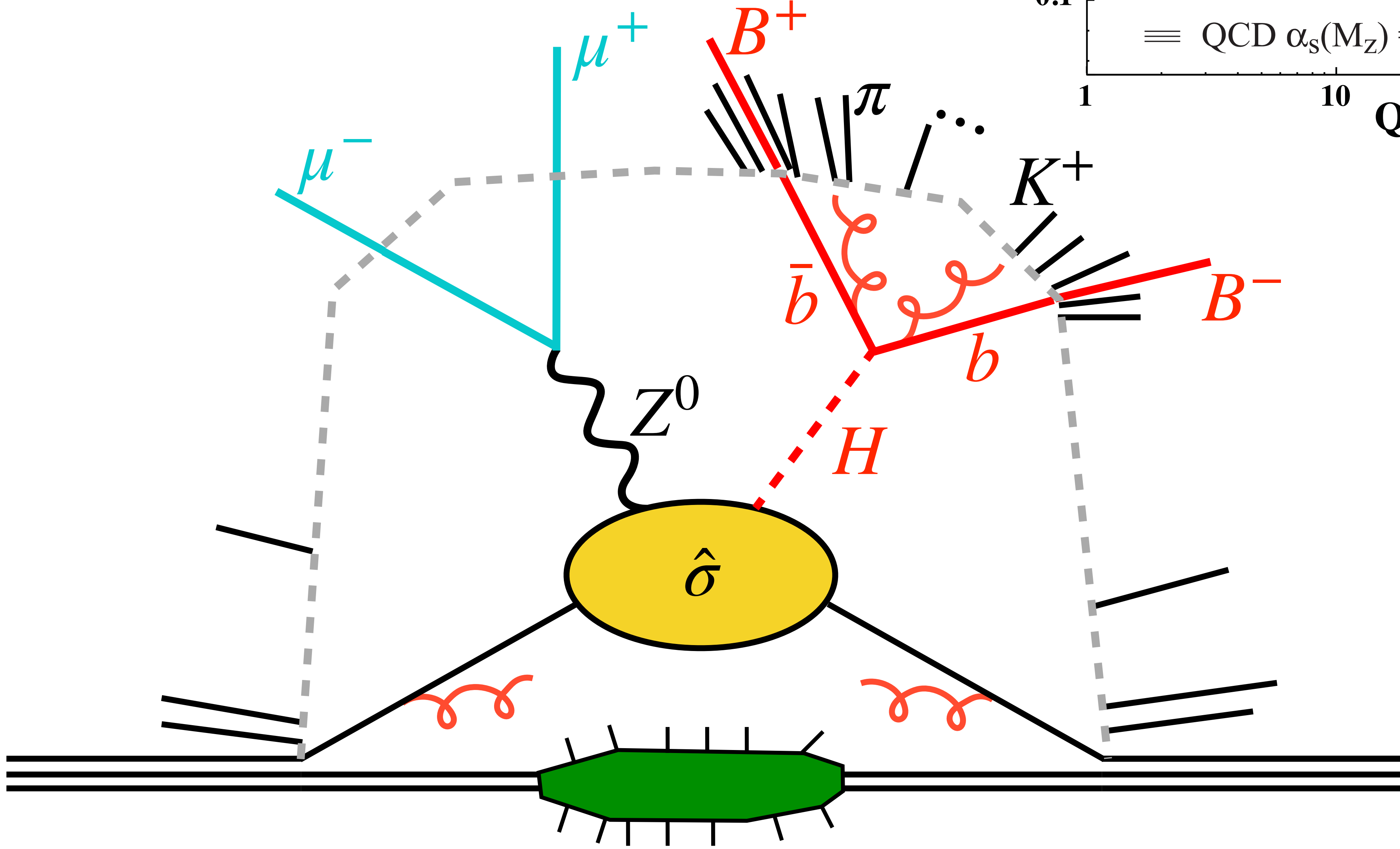
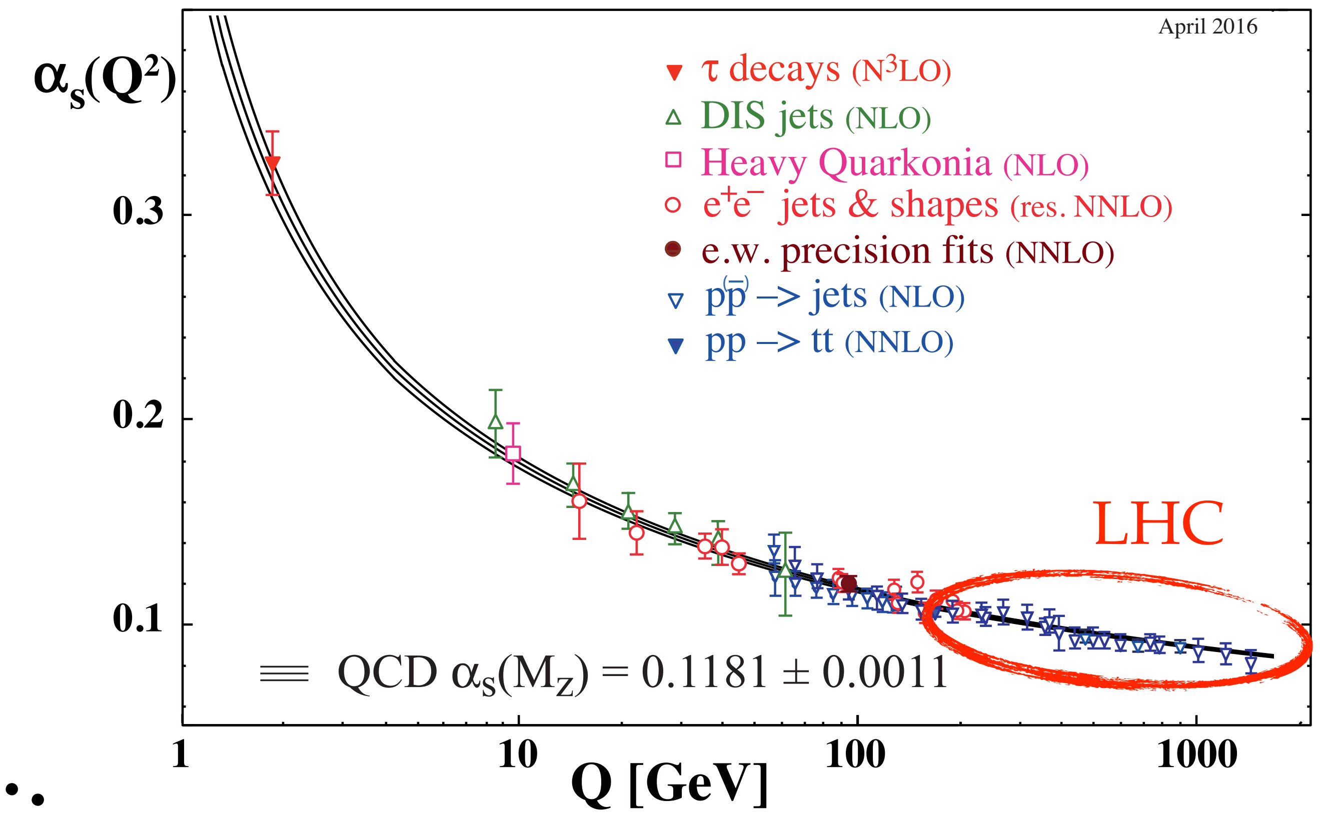


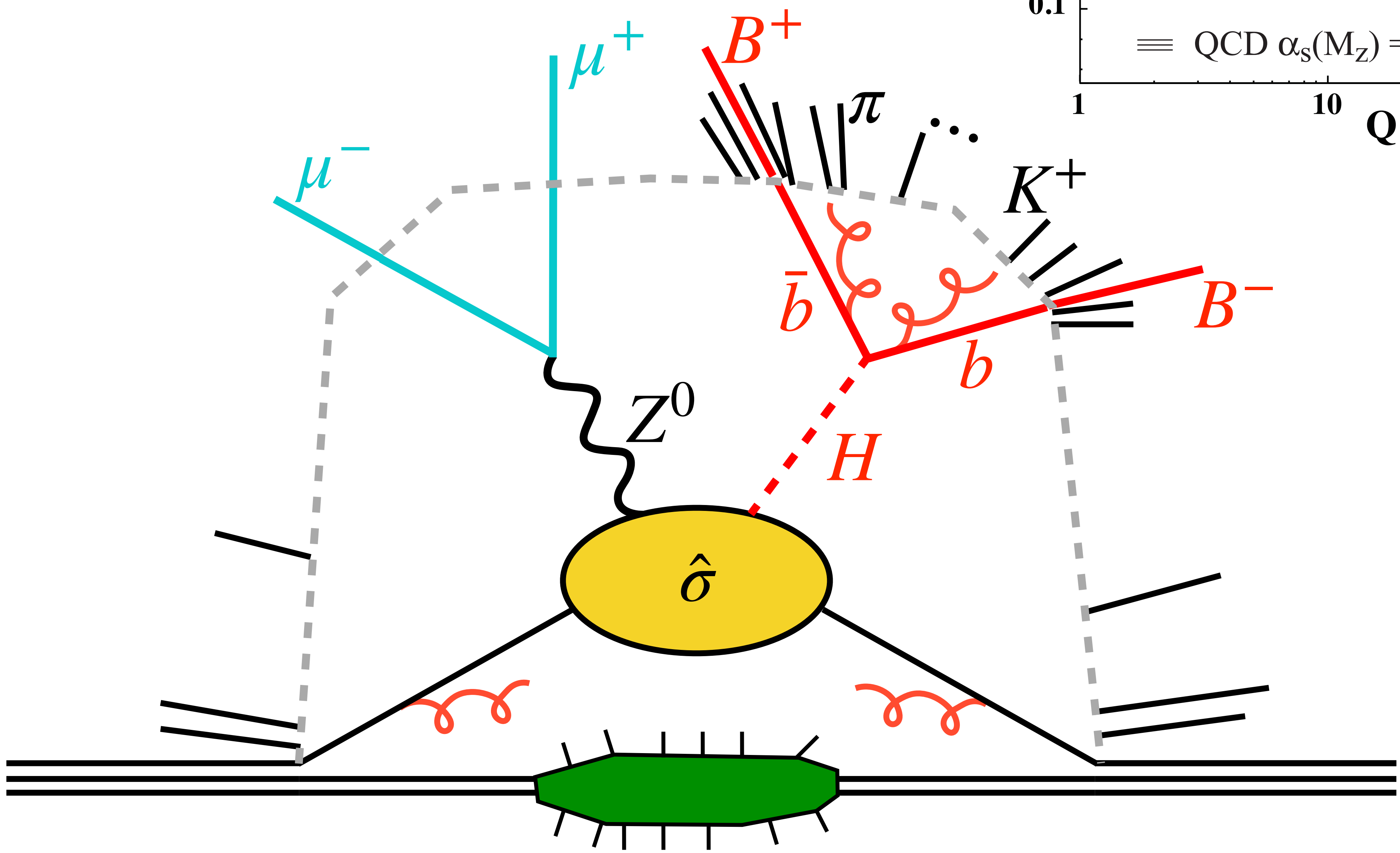


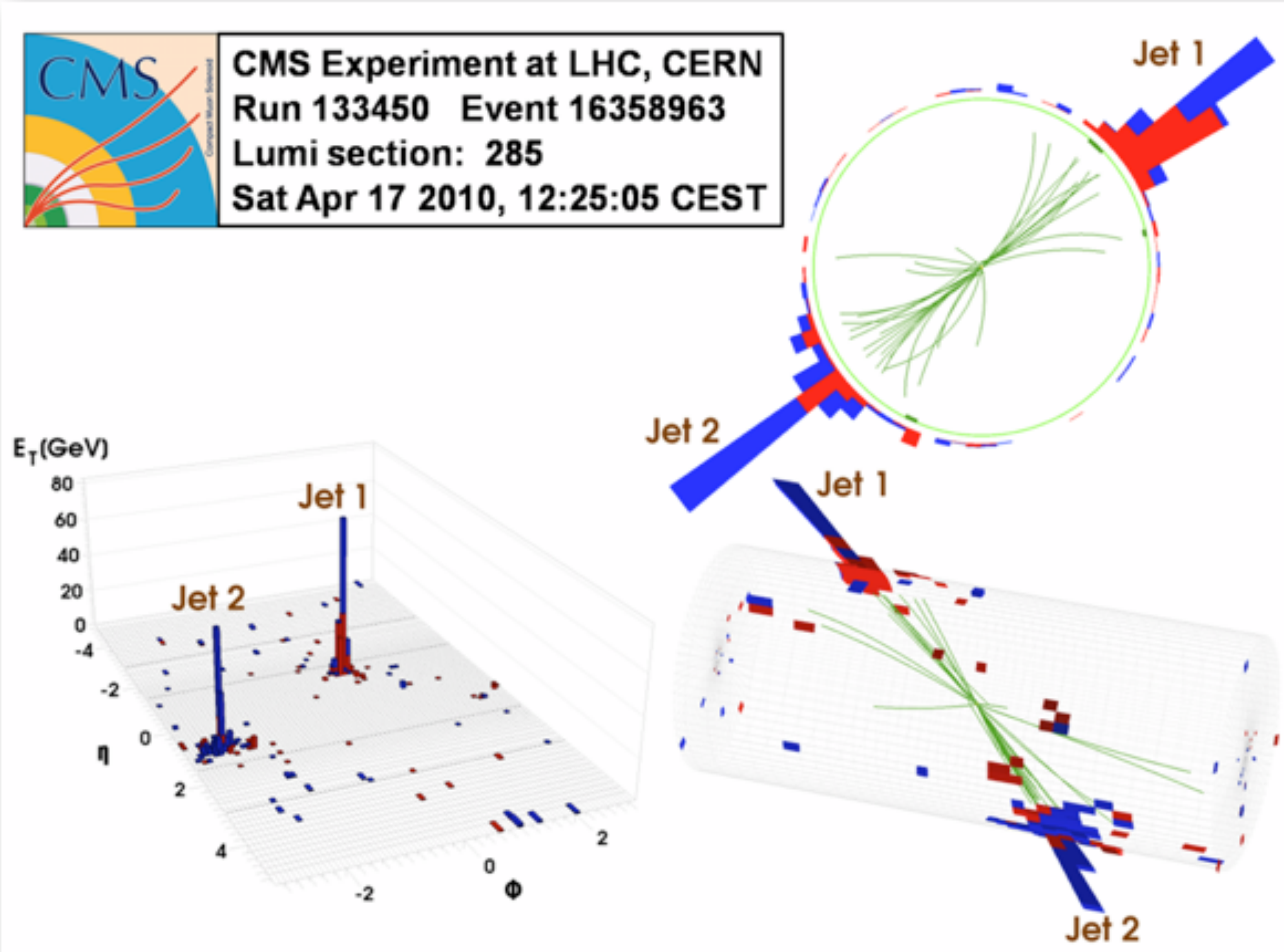
Today: Jets
20 Feb (Tue wk 6): PDFs

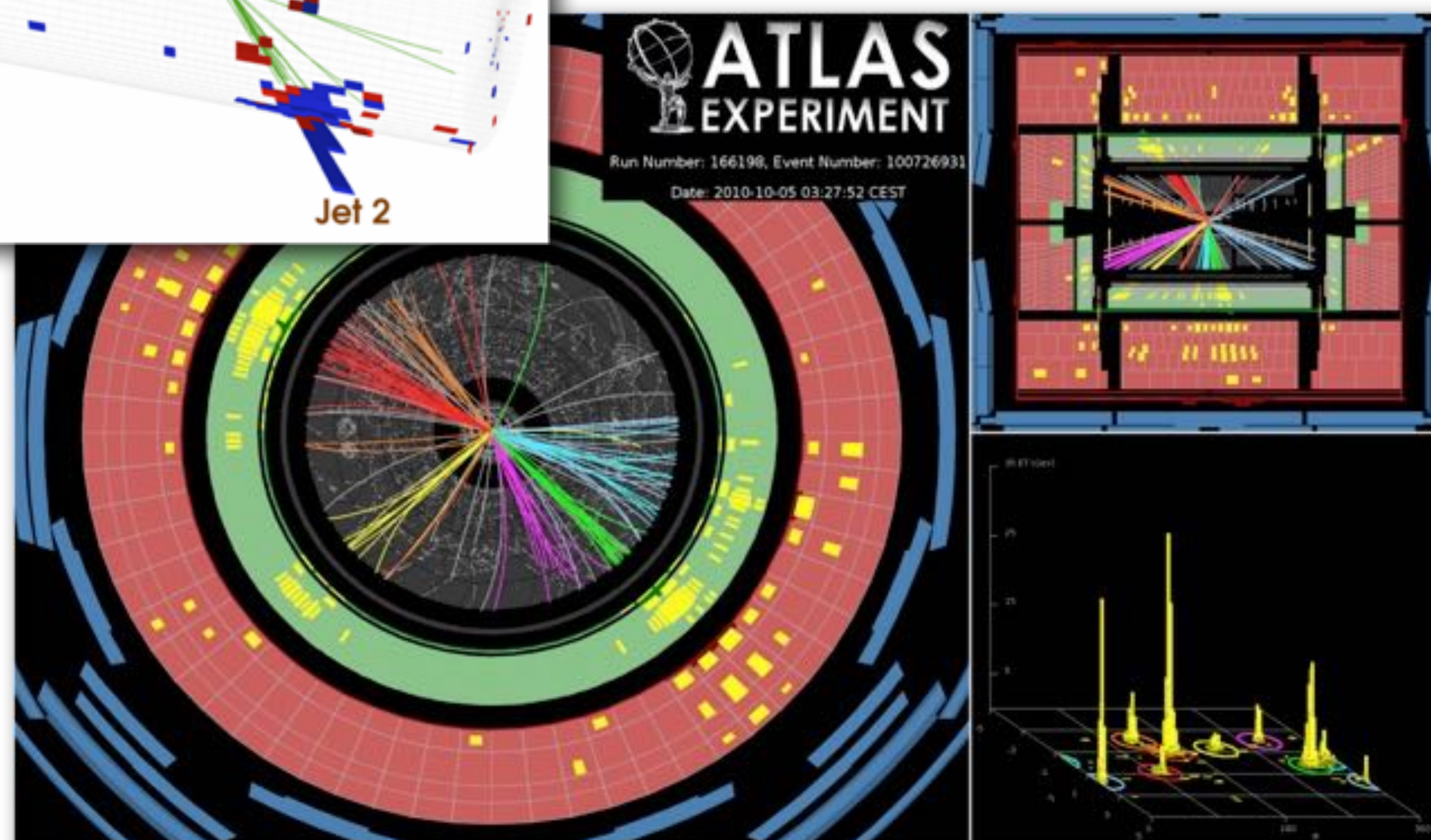
Weeks 7 & 8: QCD simulation (Helliwell, Skands)

strong coupling v. Q





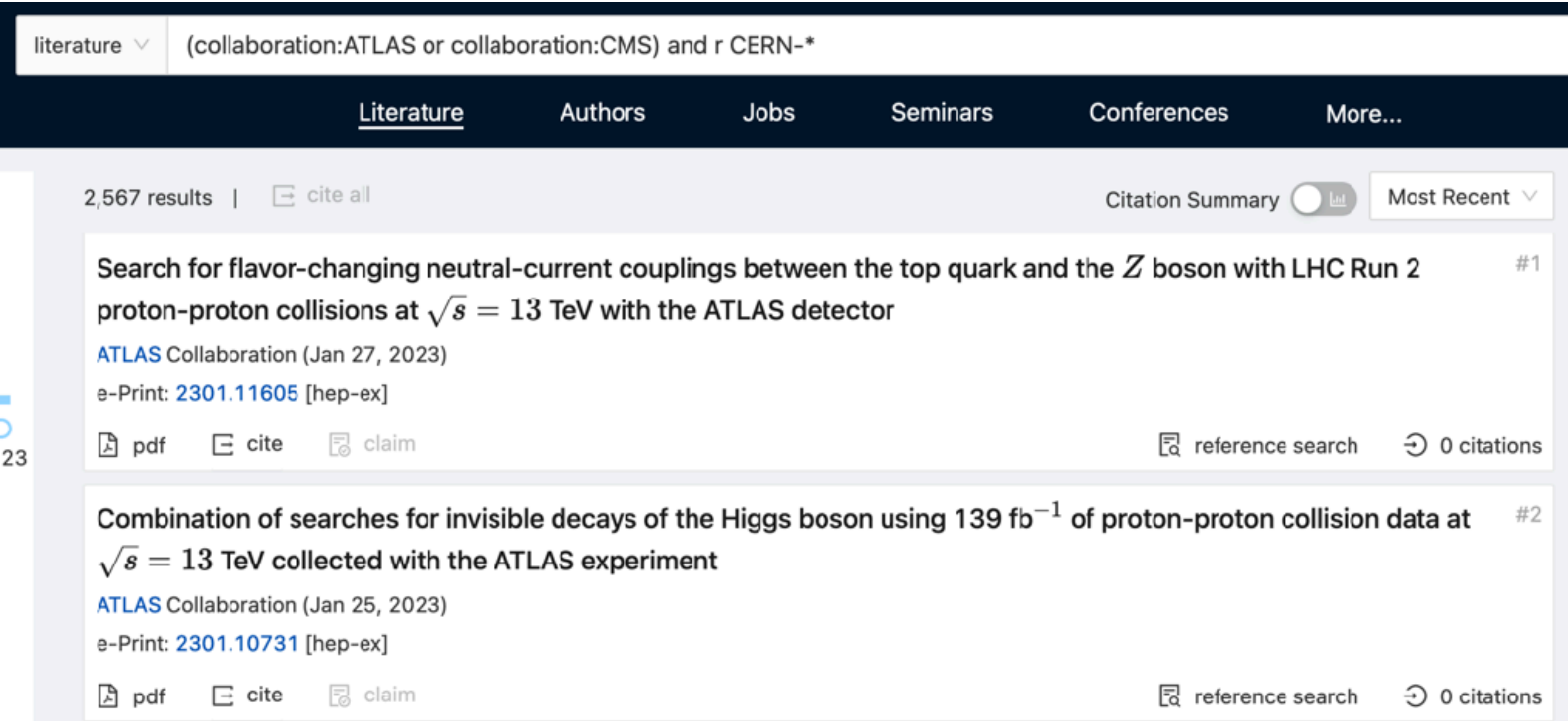




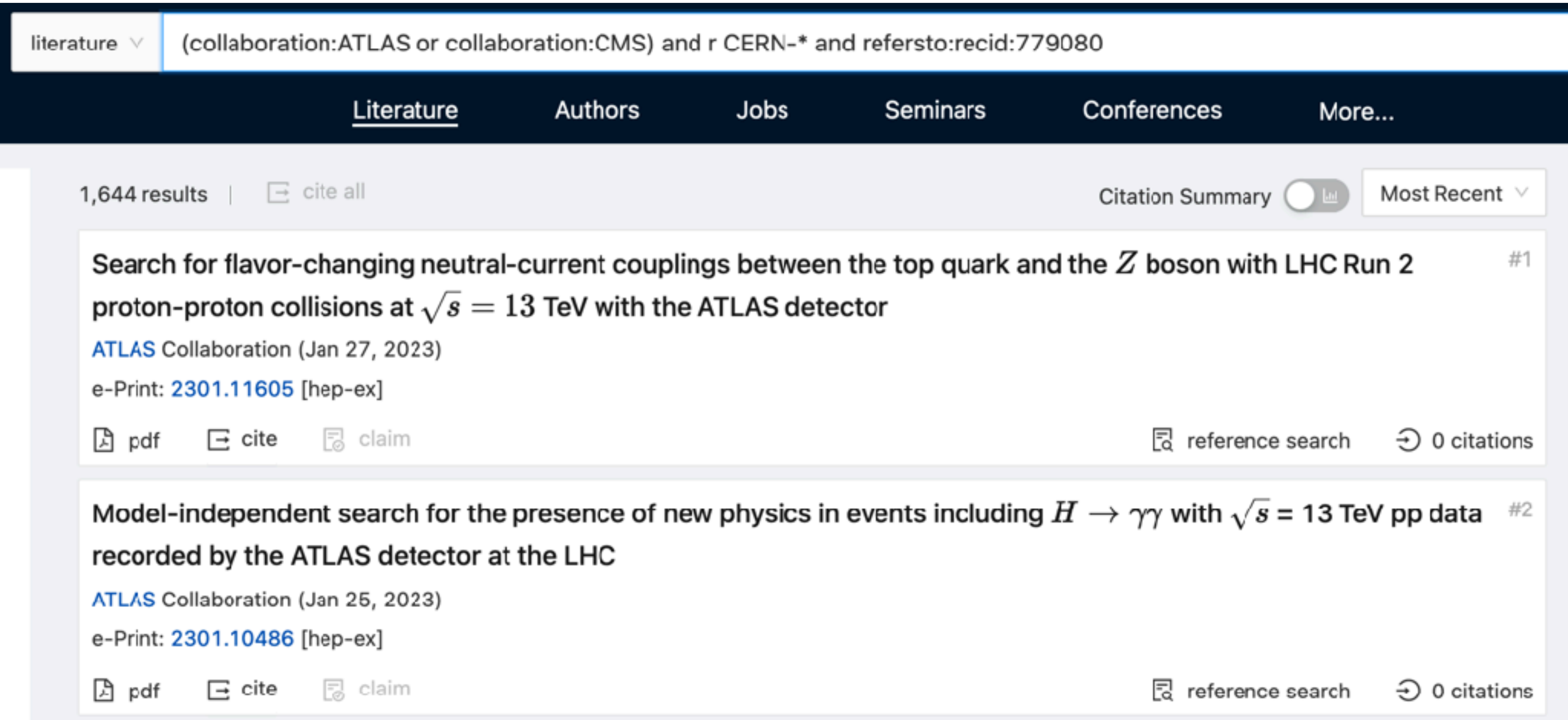
JETS

Collimated, energetic bunches of particles

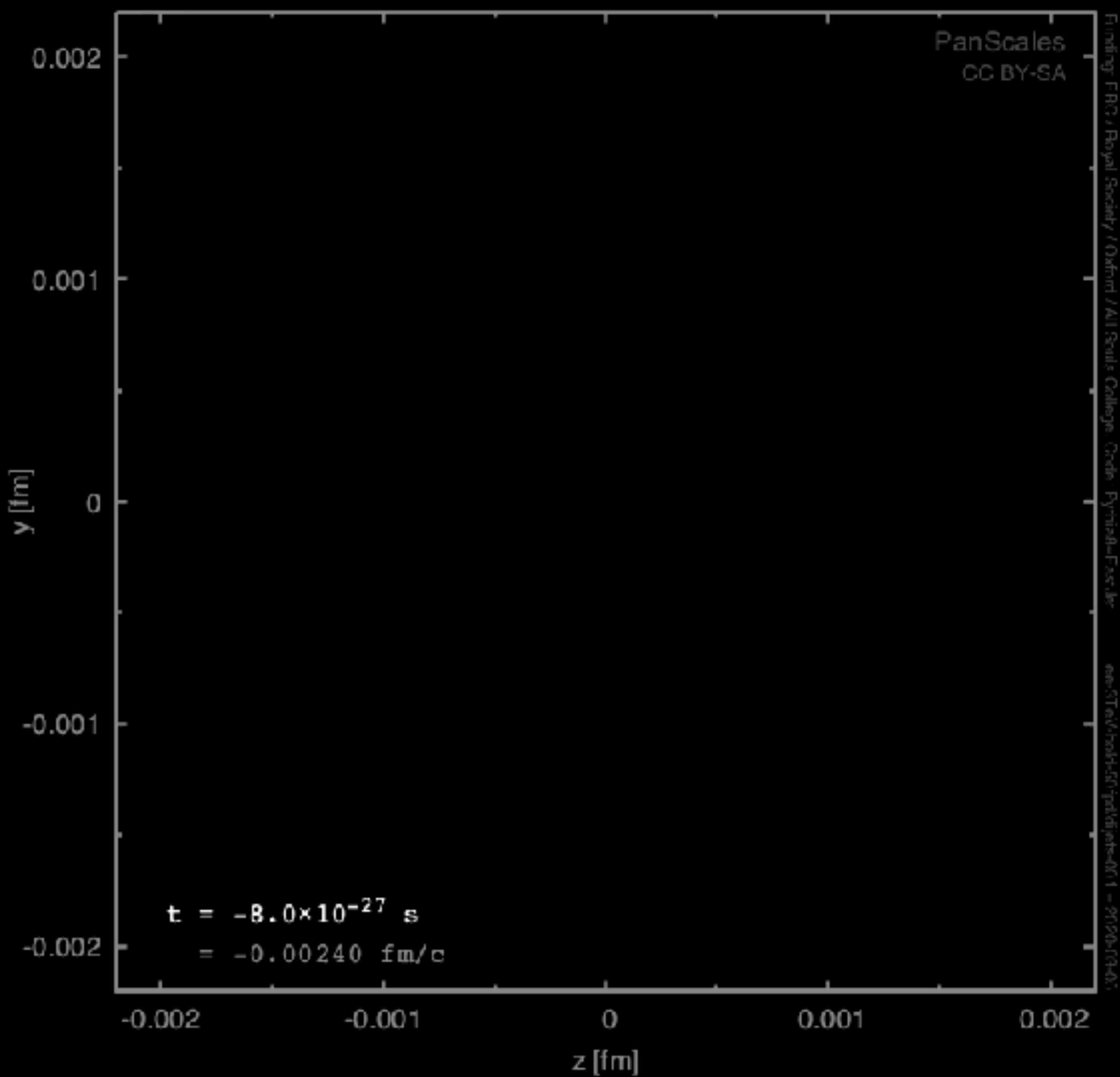
Find all papers by ATLAS and CMS
2598 records found



Pull out those that refer to one widely used jet-alg
1756 records found



$>$ 60% of papers use jets!



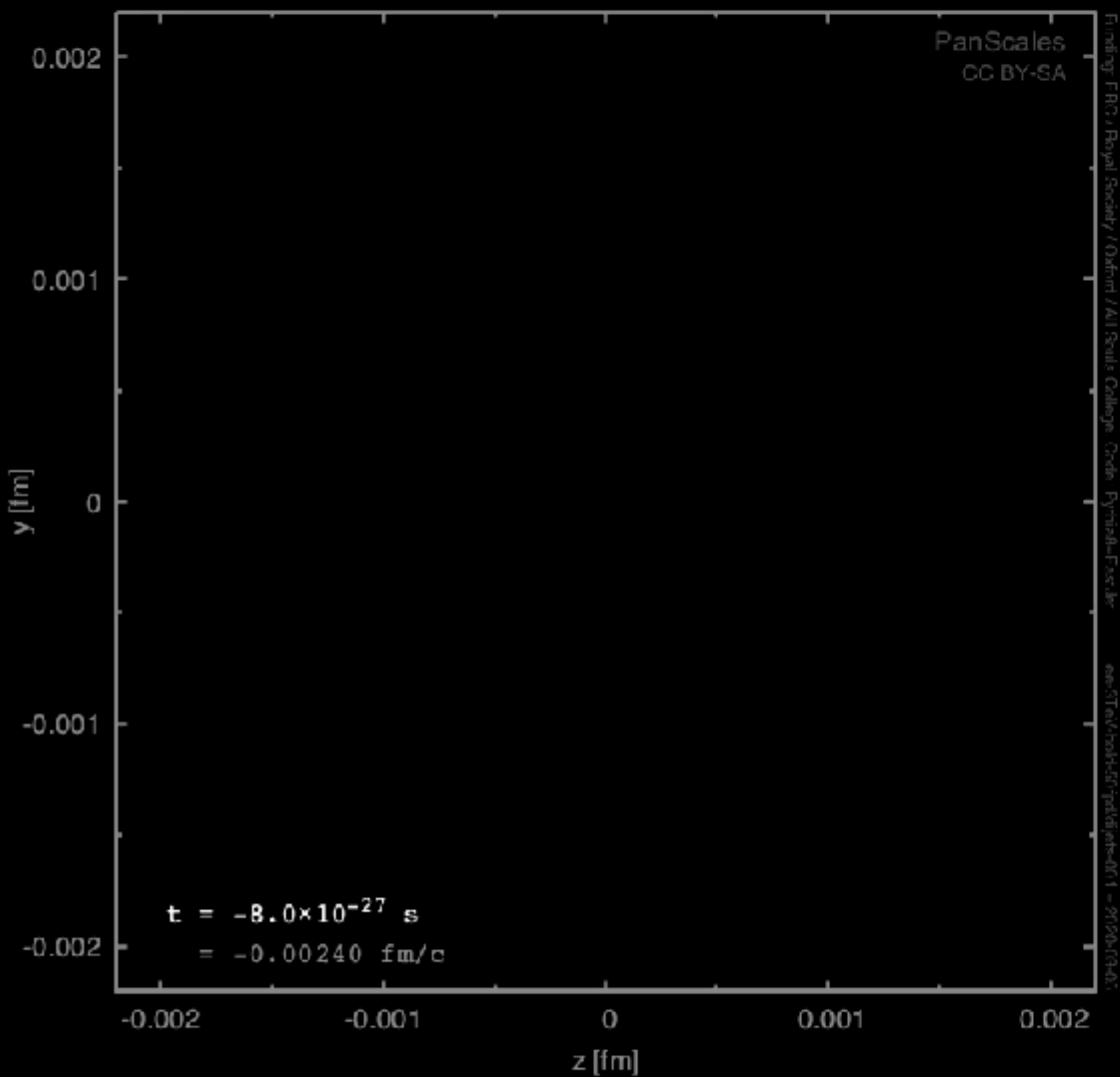
incoming beam particle

intermediate particle (quark or gluon)

final particle (hadron)

Event evolution spans 7 orders of magnitude in space-time

http://panscales.org/videos.html



incoming beam particle

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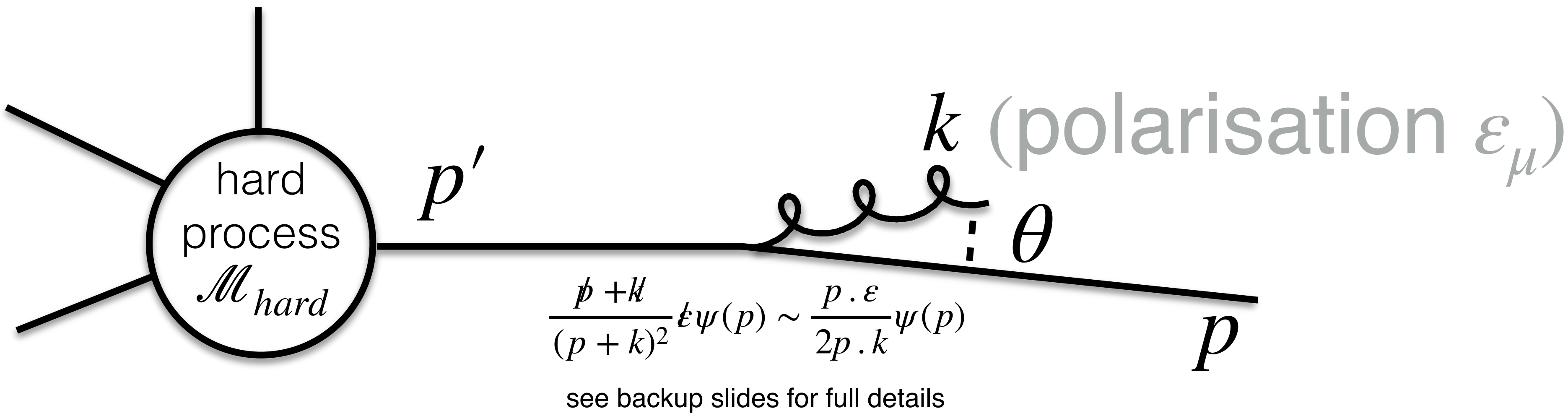
why do quarks and gluons fragment into jets?



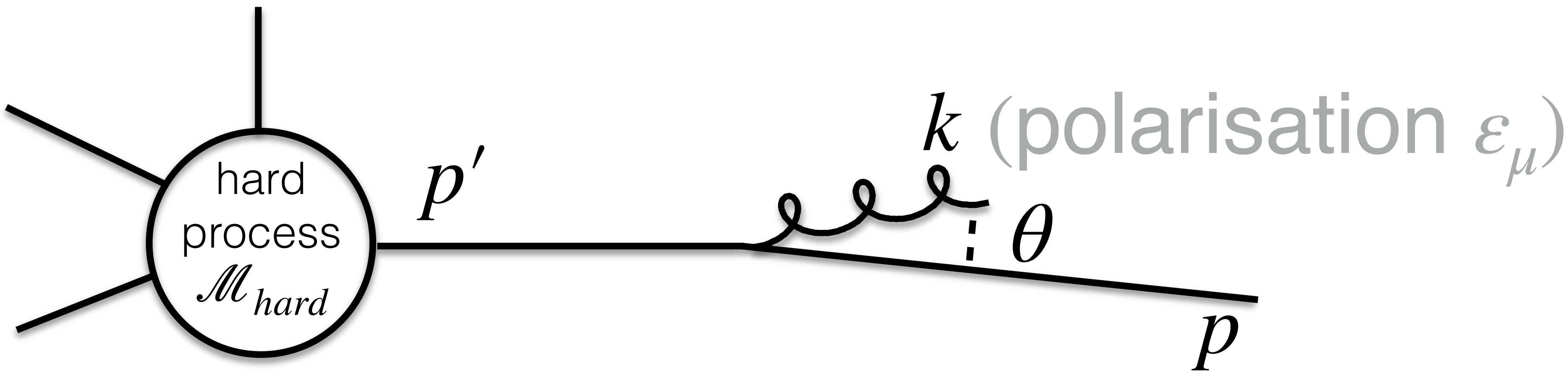
- **soft limit**: low-energy gluon emission: $E_k \ll E_p$
- **collinear limit**: small-angle emission, $\theta \ll 1$
- amplitude: (leaving out colour factors)

$$\mathcal{M} \propto \mathcal{M}_{hard} \cdot g_s \frac{p.\varepsilon}{p.k} \simeq \mathcal{M}_{hard} \cdot g_s \frac{\sin\theta}{E_k(1-\cos\theta)} \simeq \mathcal{M}_{hard} \cdot g_s \frac{2}{E_k \theta}$$

- phase space:

$$d\Phi \simeq d\Phi_{hard} \frac{E_k^2 dE_k}{(2\pi)^3 2E_k} d(\cos\theta)\, d\phi \simeq d\Phi_{hard} \cdot E_k dE_k \frac{\theta d\theta}{4\pi^2} \frac{d\phi}{2\pi}$$

Soft & collinear gluon emission



- Full result:

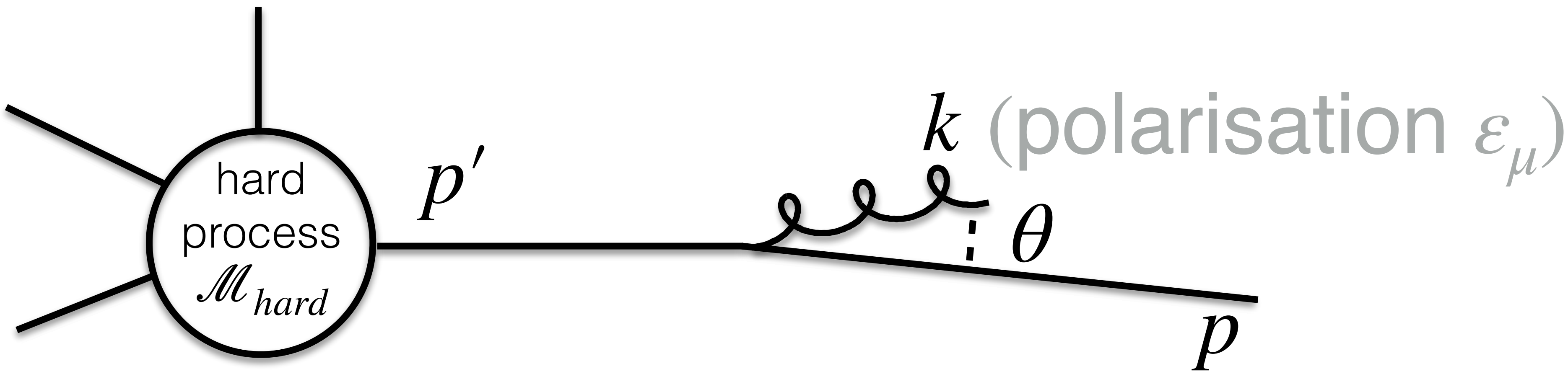
$$|\mathcal{M}|^2 d\Phi \simeq |\mathcal{M}_{hard}|^2 d\Phi_{hard} \cdot \frac{2C_F \alpha_s}{\pi} \frac{dE_k}{E_k} \frac{d\theta}{\theta}$$

- **factorises** into product of original hard matrix element and an additional soft-gluon emission probability

$$dP_{soft-gluon-emission} = \frac{2C_F \alpha_s}{\pi} \frac{dE_k}{E_k} \frac{d\theta}{\theta}$$

- **diverges** in soft ($E_k \to 0$) and collinear ($\theta \to 0$) limits

total probability of gluon emission



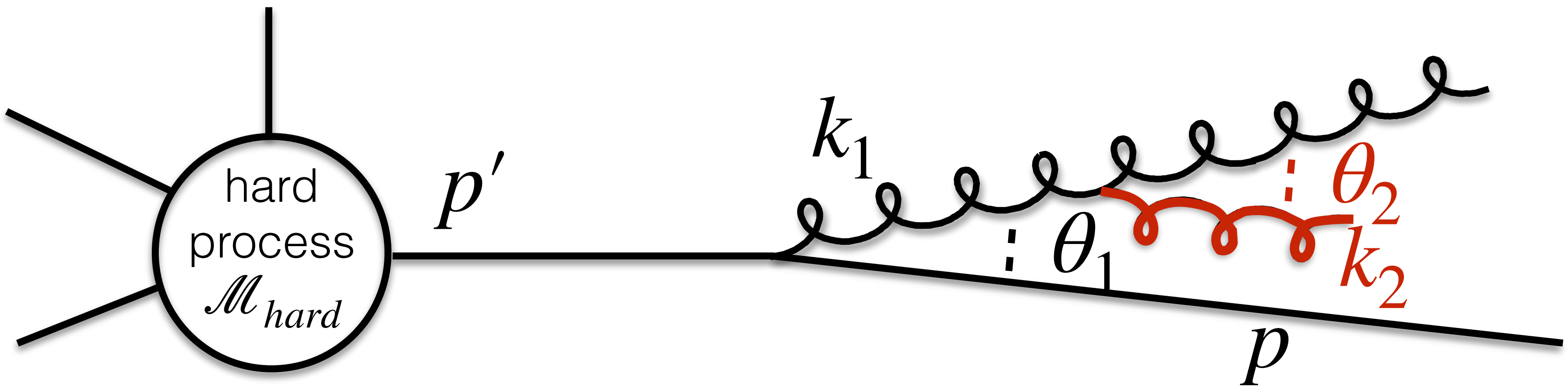
Total probability of gluon emission:

$$\langle N_{gluon}\rangle \simeq P_{gluon-emission} = \int dP = \frac{2\alpha_s C_F}{\pi}\int_{\Lambda_{QCD}/E_p}^{1}\frac{d\theta}{\theta}\int_{\Lambda_{QCD}/\theta}^{E_p}\frac{dE_k}{E_k} = \frac{\alpha_s C_F}{\pi}\ln^2\frac{E_p}{\Lambda_{QCD}}$$

Suppose (~wrongly!) that scale of the coupling is given by E_p, i.e. use $\alpha_s(E_p) = (2b_0 \ln E_p/\Lambda_{QCD})^{-1}$, then

$$\langle N_{gluon}\rangle \simeq P_{gluon-emission} \simeq \frac{C_F}{\pi b_0}\ln\frac{E_p}{\Lambda_{QCD}} \sim \frac{1}{\alpha_s} \gg 1$$

i.e. gluon emission is bound to happen and the average number of gluons is large



Emission of gluon 1 from gluon 2 also **factorises,** with colour factor $C_A = 3$, instead of $C_F = 4/3$:

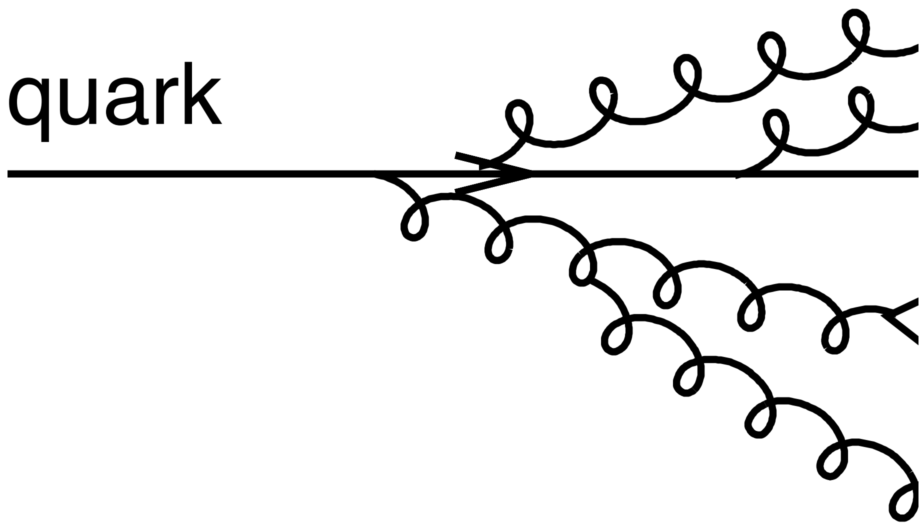
$$dP_{gluon-2-from-gluon-1} = \frac{2C_A \alpha_s}{\pi} \frac{dE_2}{E_2} \frac{d\theta_2}{\theta_2}$$

Additional gluon radiation due to emission from gluon 1 is confined within a cone of angle $\theta_2 \lesssim \theta_1$ (a.k.a. **Angular Ordering**) and will have energy $E_2 \lesssim E_1$

Total extra number of gluons emitted from gluon 1:

$$\frac{\alpha_s C_A}{\pi} \ln^2 \frac{E_1 \theta_1}{\Lambda_{QCD}}$$

Why do we see jets?

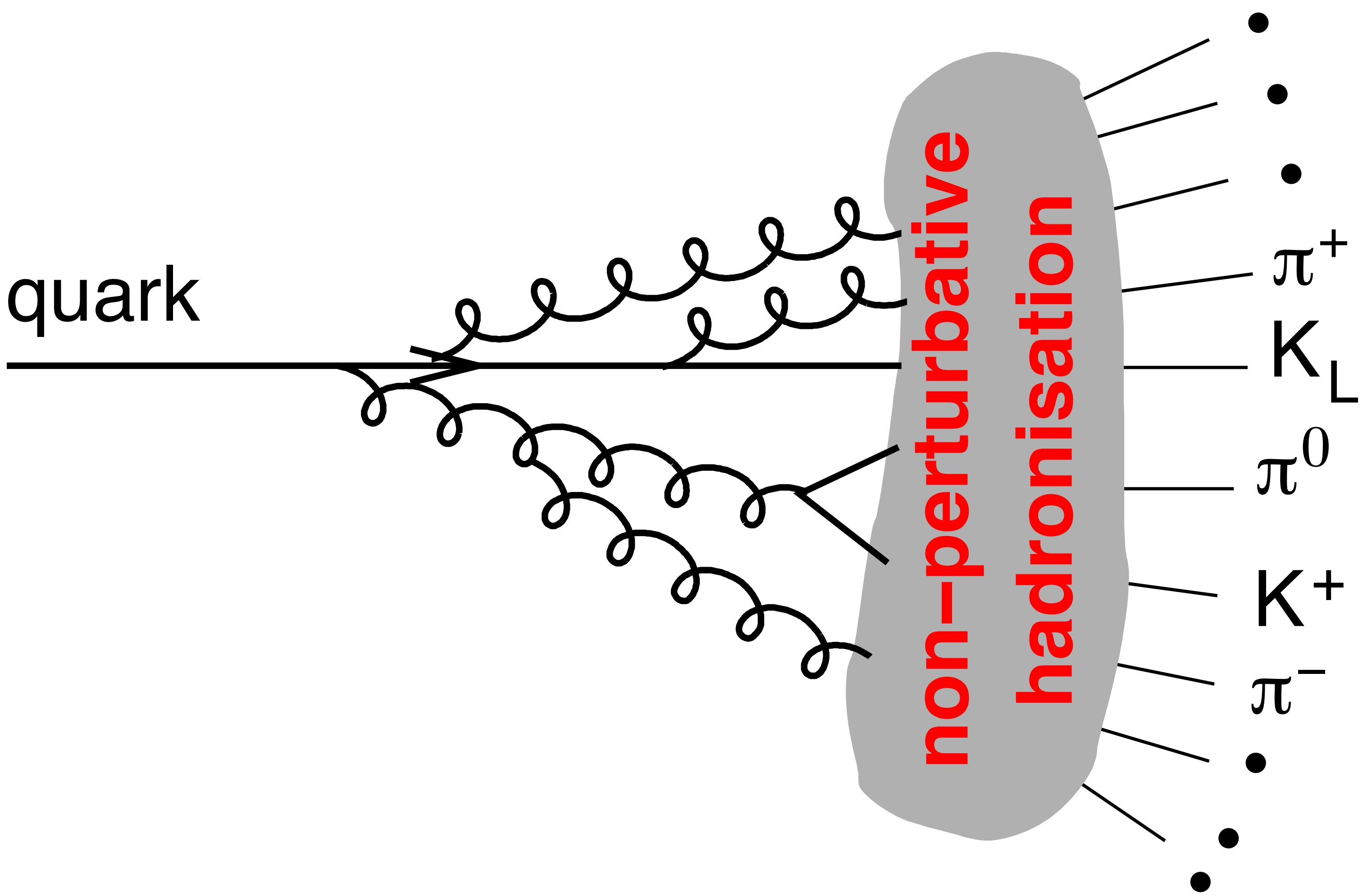


Starting from energetic quark, emit a cascade of many low-energy (soft) and small-angle (collinear) gluons

$$\frac{\alpha_s C_{F/A}}{\pi} \frac{dE}{E} \frac{d\theta}{\theta}$$

giving a collimated jet of partons

Why do we see jets?



Starting from energetic quark, emit a cascade of many low-energy (soft) and small-angle (collinear) gluons

$$\frac{\alpha_s C_{F/A}}{\pi} \frac{dE}{E} \frac{d\theta}{\theta}$$

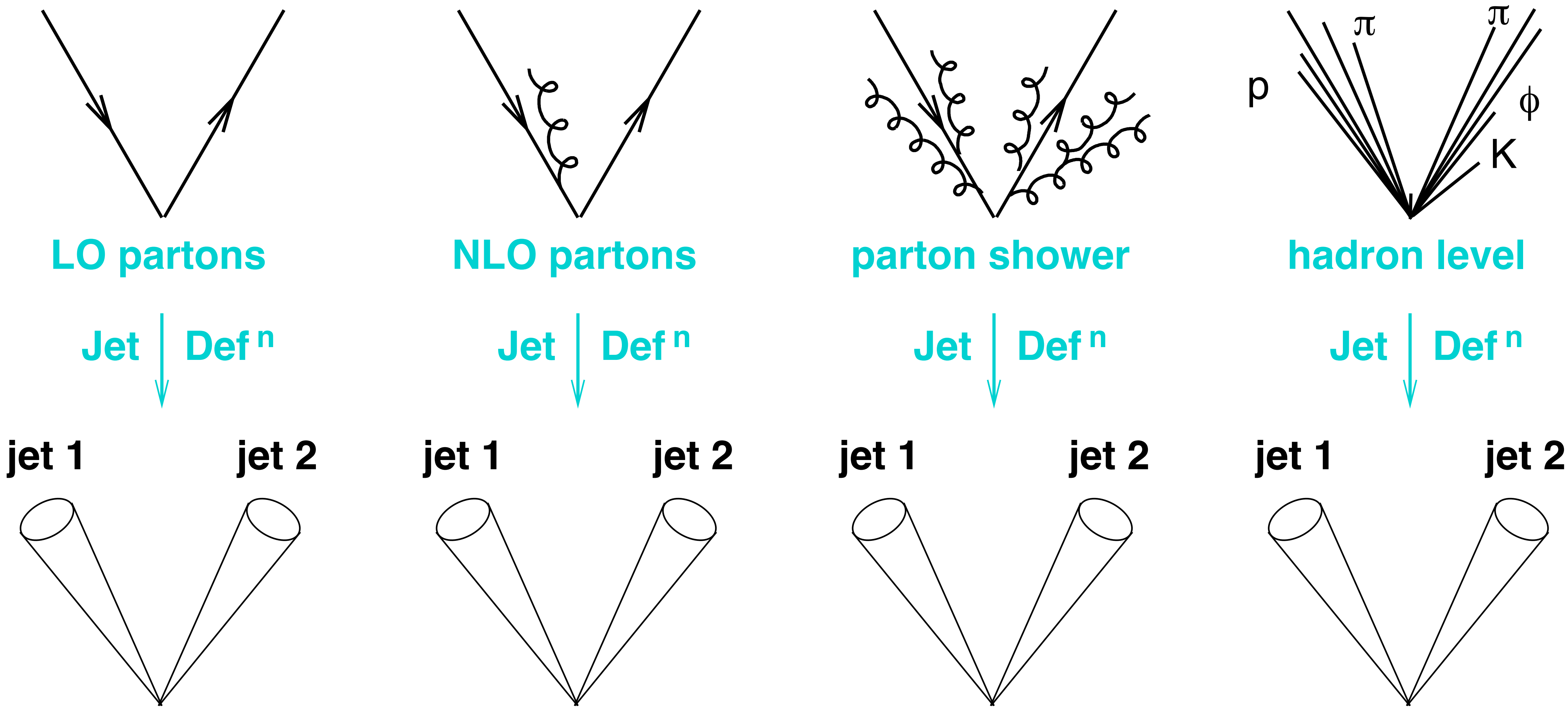
giving a collimated jet of partons (mostly gluons)

When the partons become separated by a distance $\sim 1/\Lambda_{QCD}$ they **confine** into hadrons.

The hadrons go in similar directions to the partons

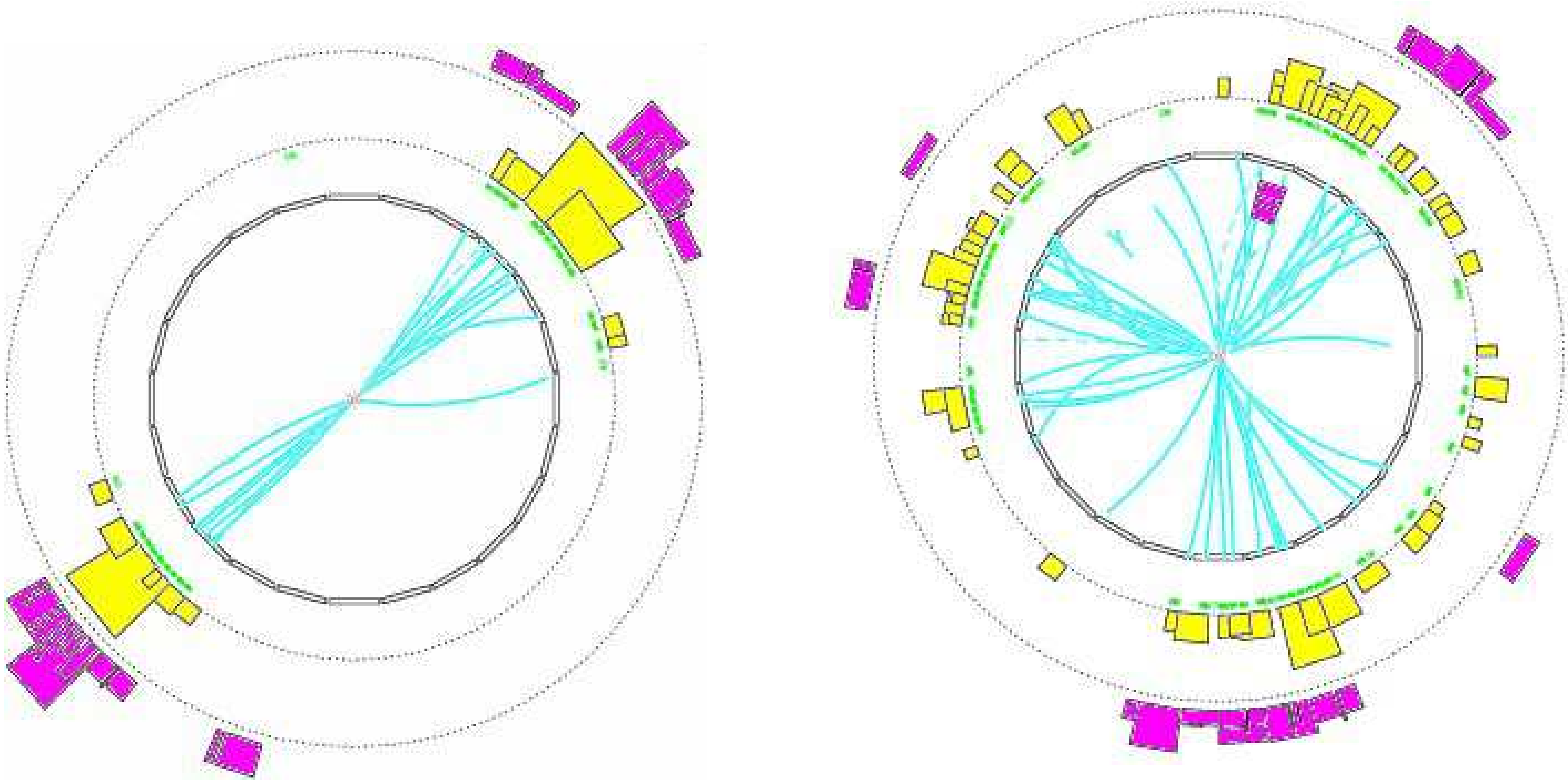
core ideas in jet reconstruction

Jet finding as a form of projection



Projection to jets should be resilient to QCD effects

Reconstructing jets is an ambiguous task



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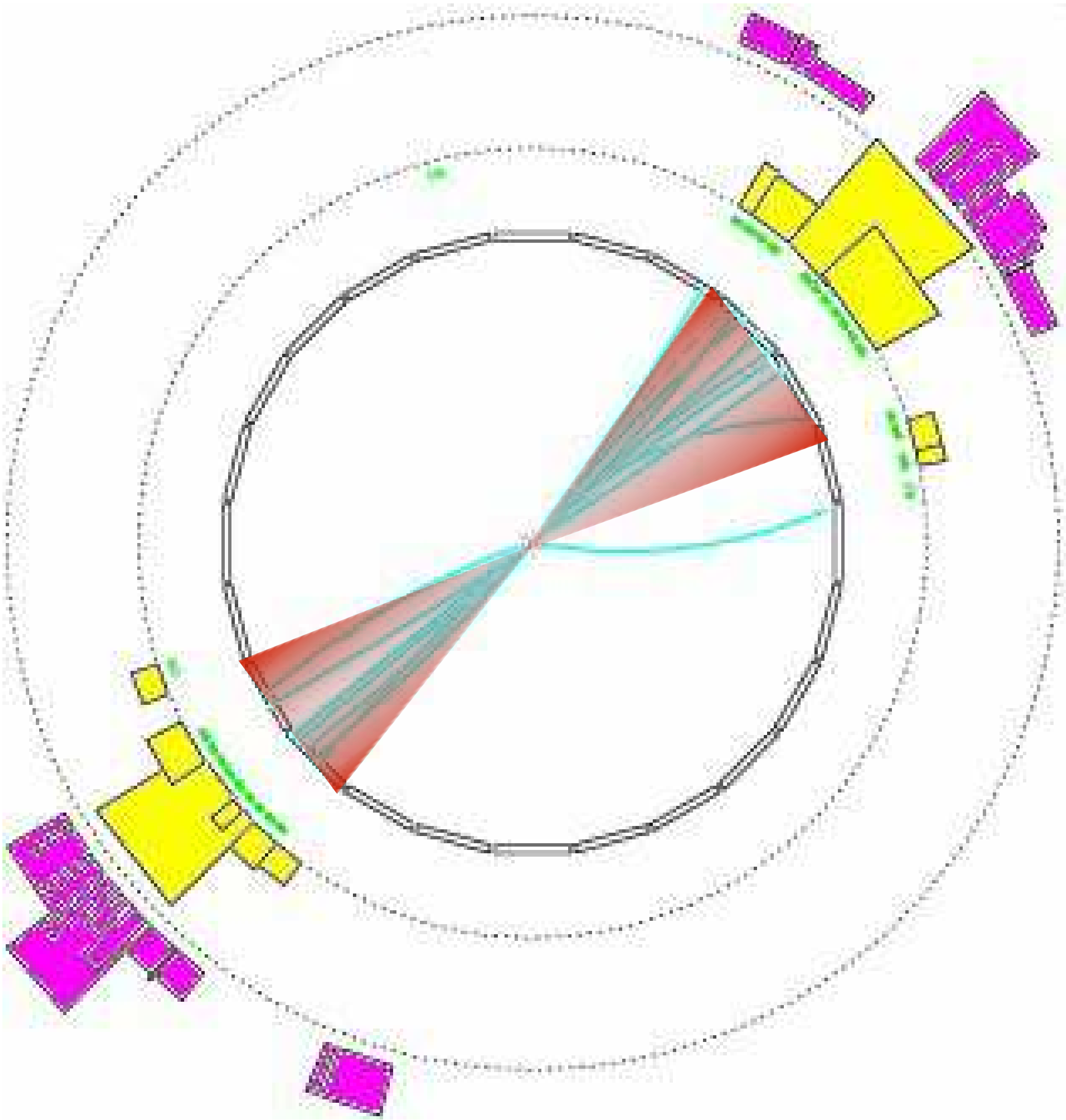
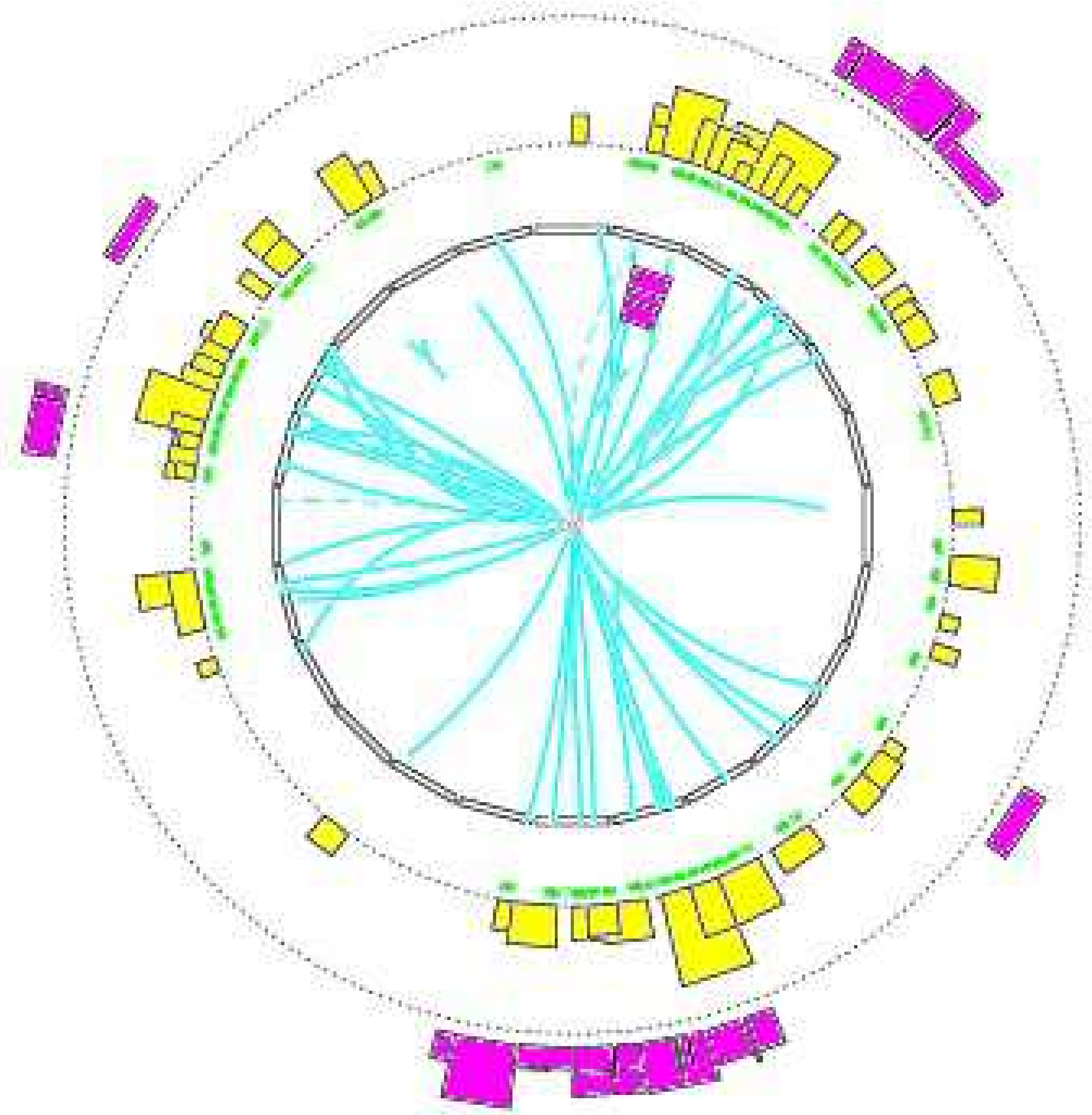
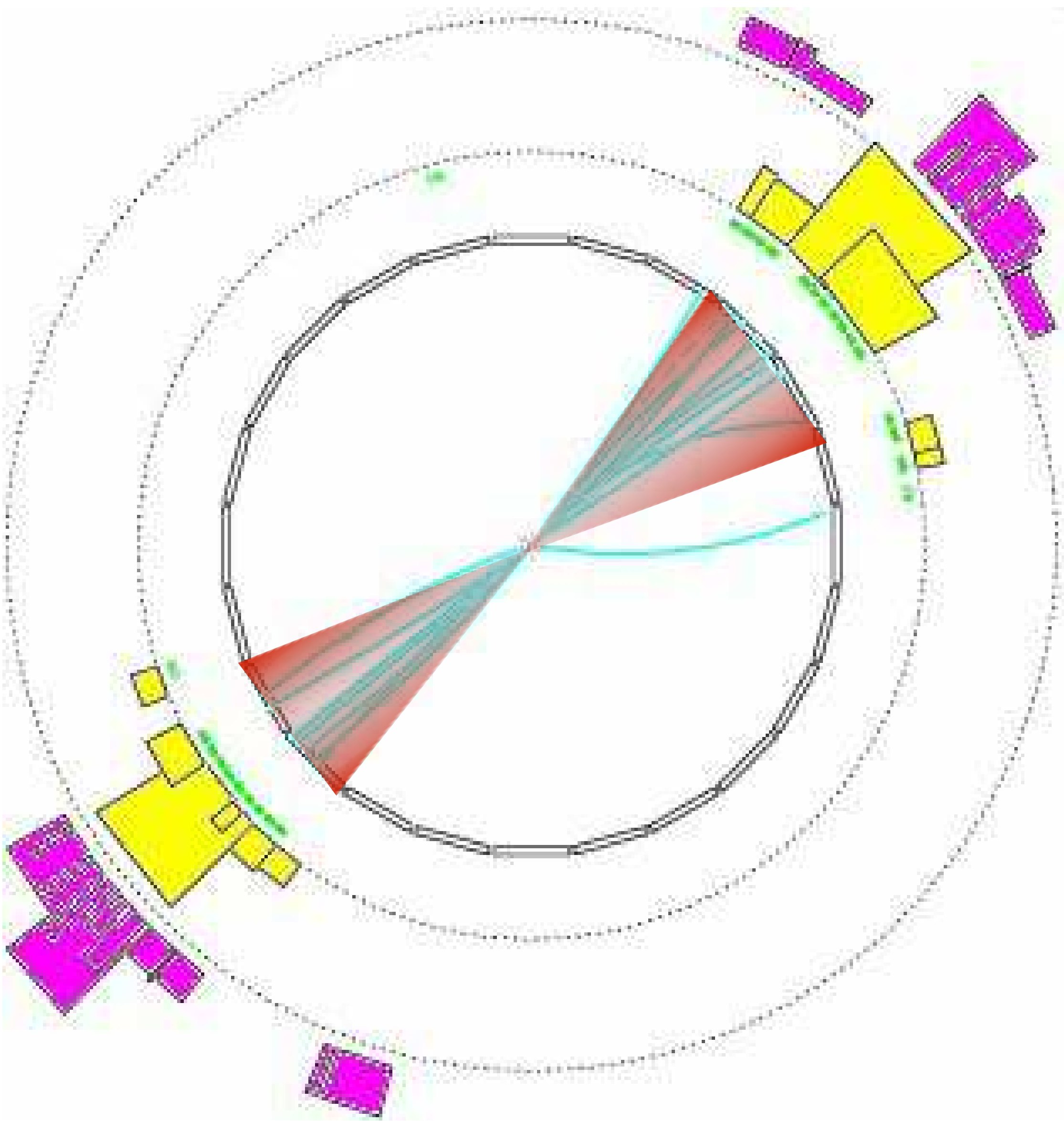




2 clear jets

Reconstructing jets is an ambiguous task



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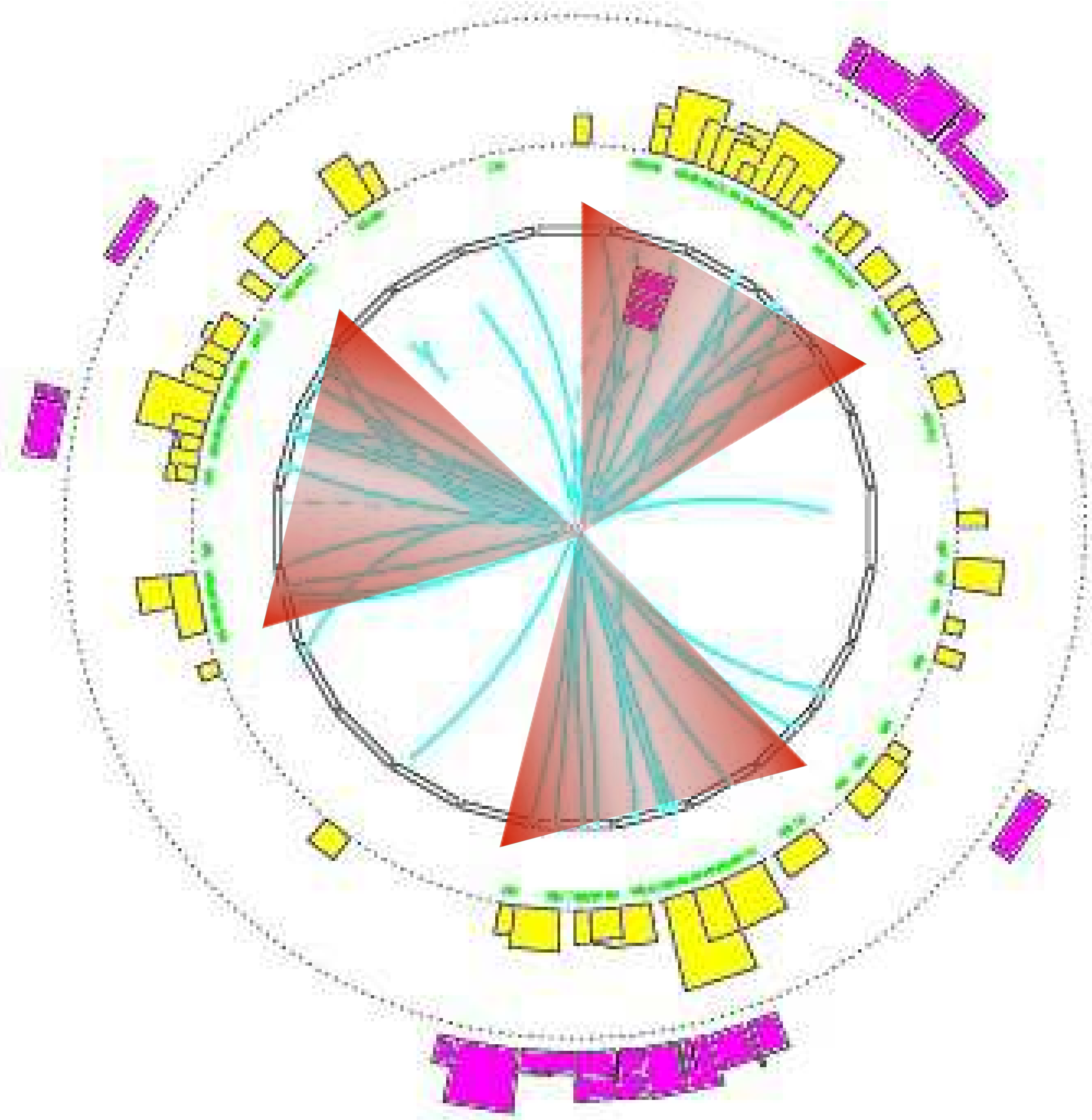
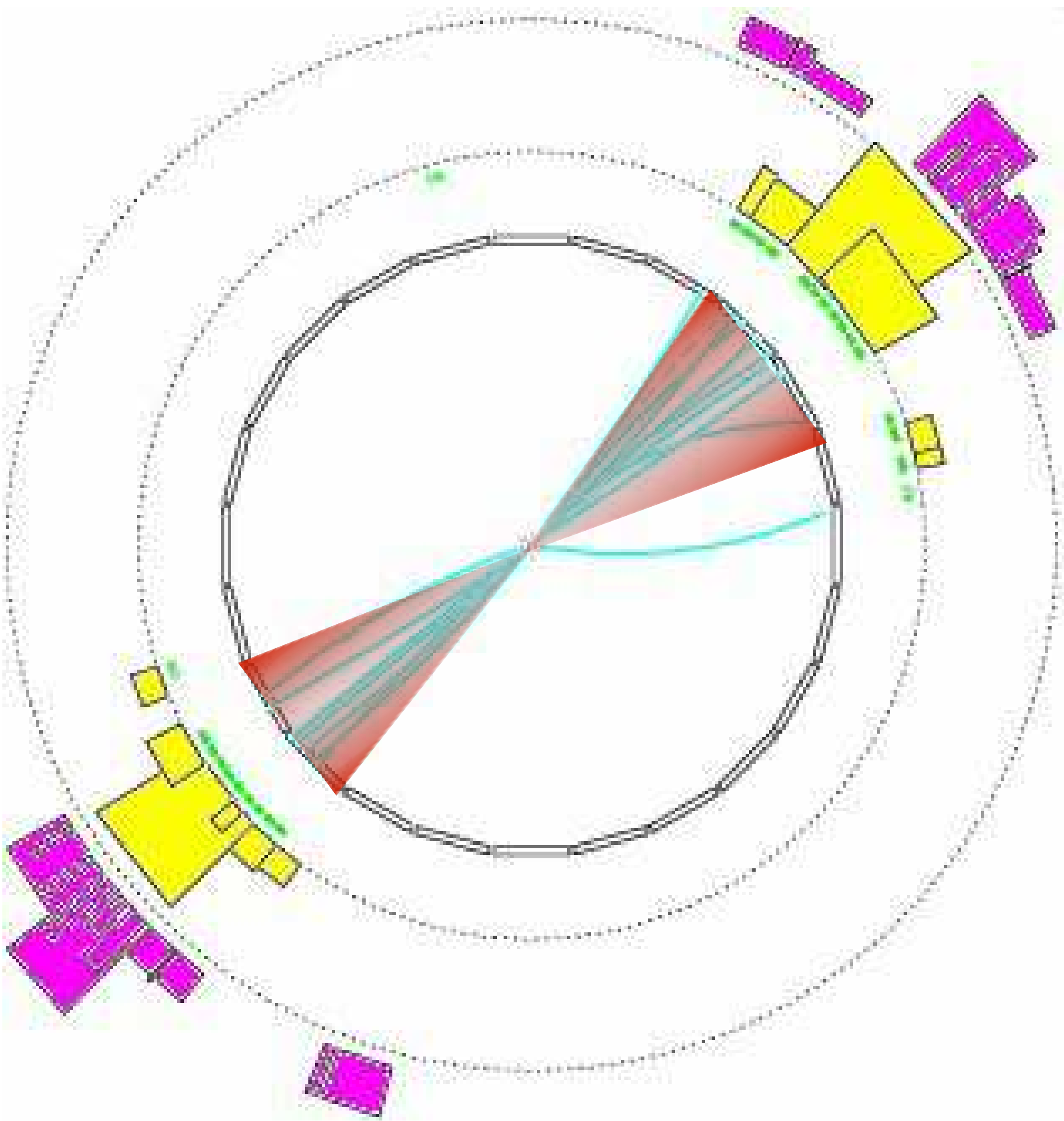
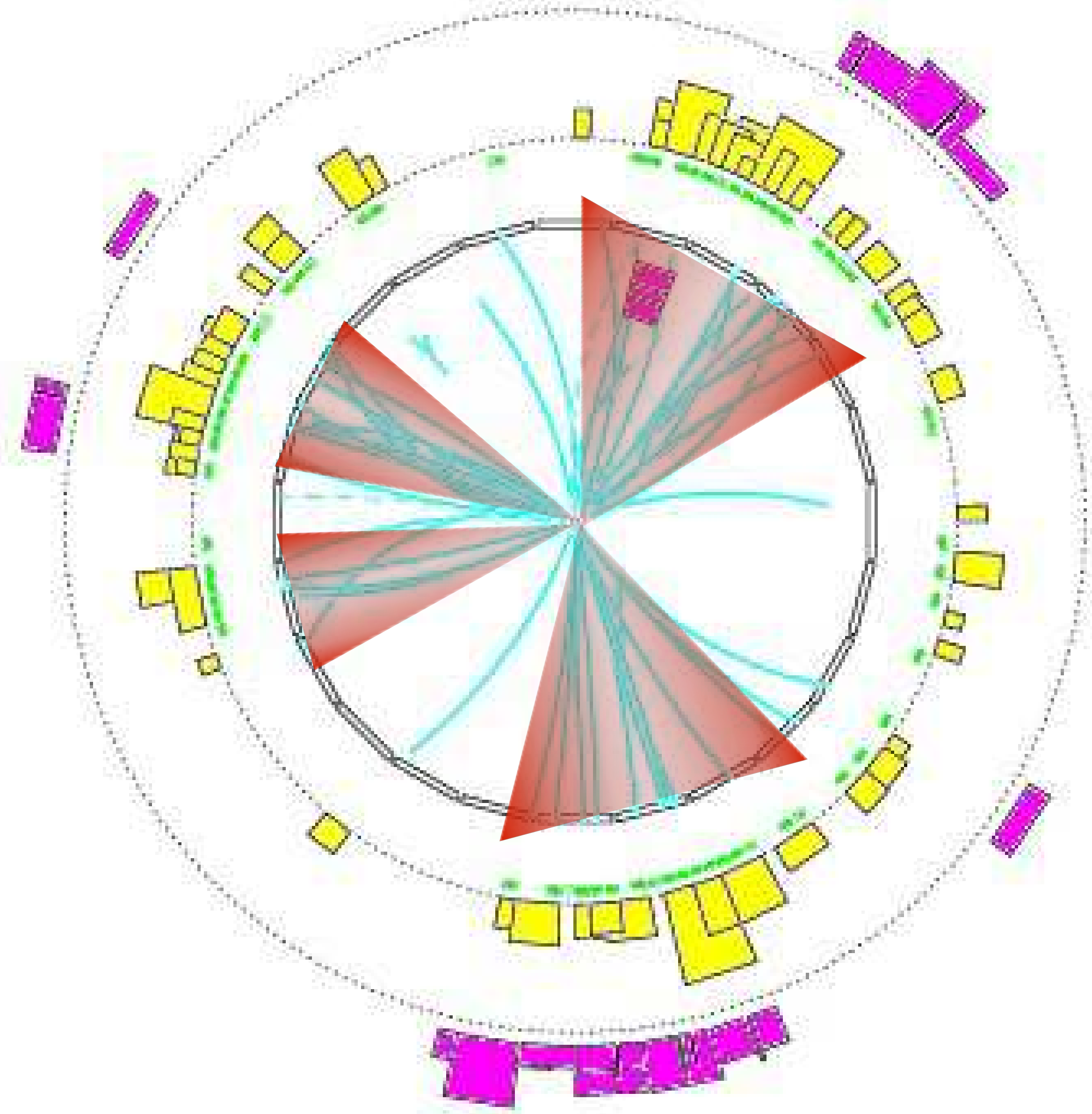


3 jets?

Reconstructing jets is an ambiguous task

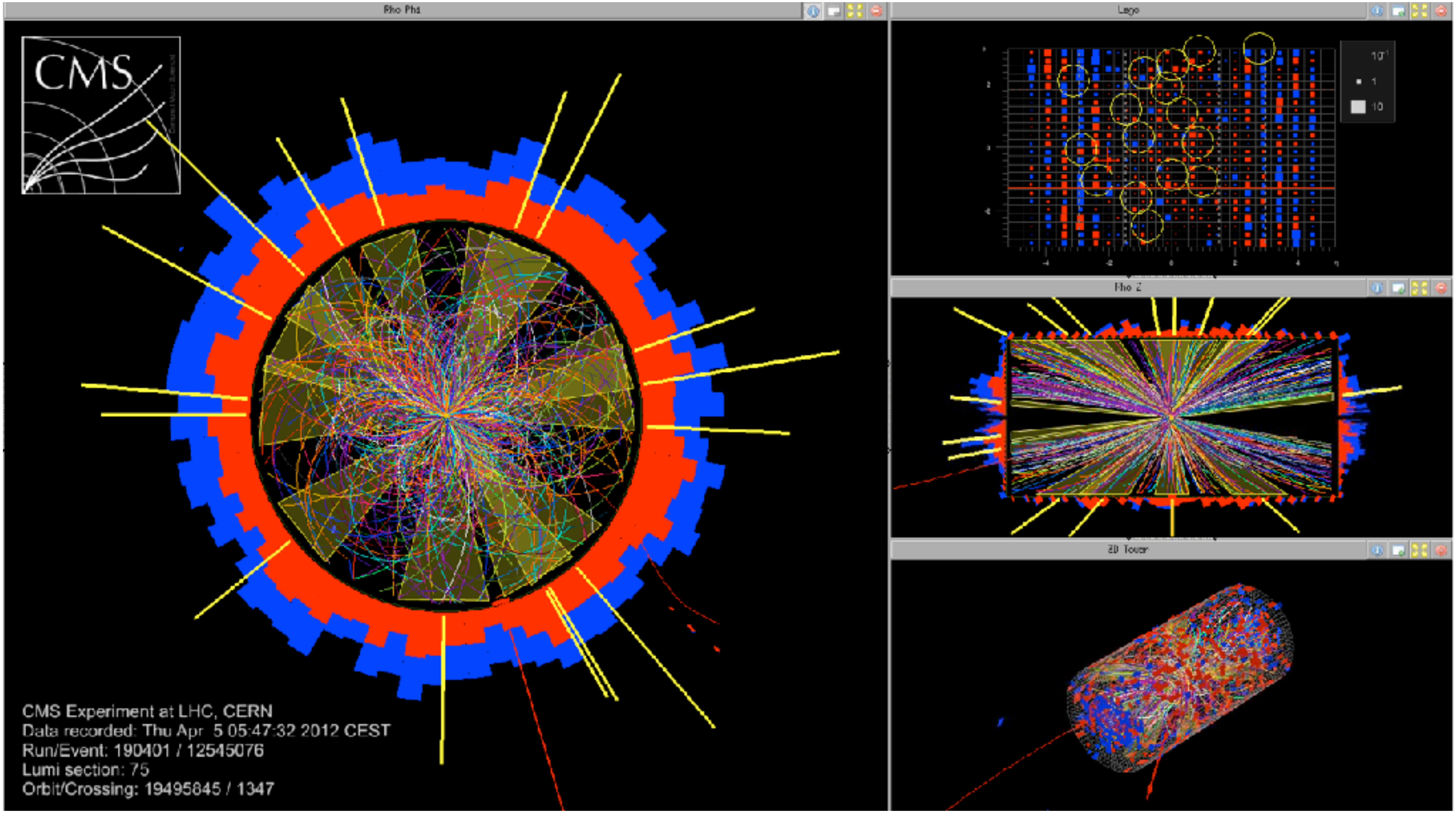


2 clear jets



3 jets?

or 4 jets?


Rho Phi
CMS
CMS Experiment at LHC, CERN
Data recorded: Thu Apr 5 05:47:32 2012 CEST
Run/Event: 190401 / 12545076
Lumi section: 75
Orbit/Crossing: 19495845 / 1347
Lego
Rho Z
3D Tower

Make a choice: specify a jet definition

$$\{p_i\} \xrightarrow{\text{jet definition}} \{j_k\}$$

particles,
4-momenta,
calorimeter towers, ...

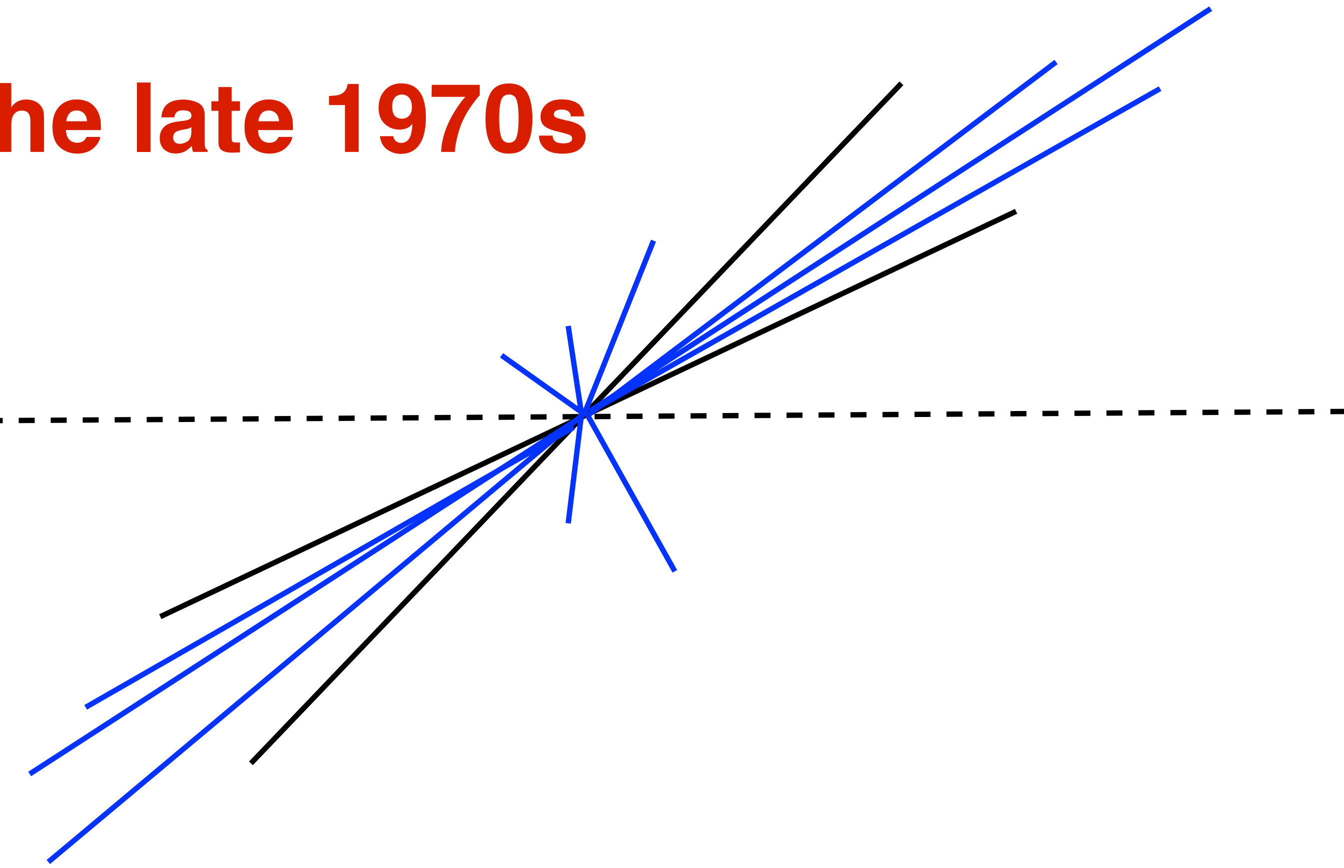
jets

- Which particles do you put together into a same jet?
- How do you recombine their momenta
(4-momentum sum is the obvious choice, right?)

"Jet [definitions] are legal contracts between theorists and experimentalists"
-- MJ Tannenbaum

They're also a way of organising the information in an event
1000's of particles per events, up to 40.000,000 events per second

Jet definitions date back to the late 1970s

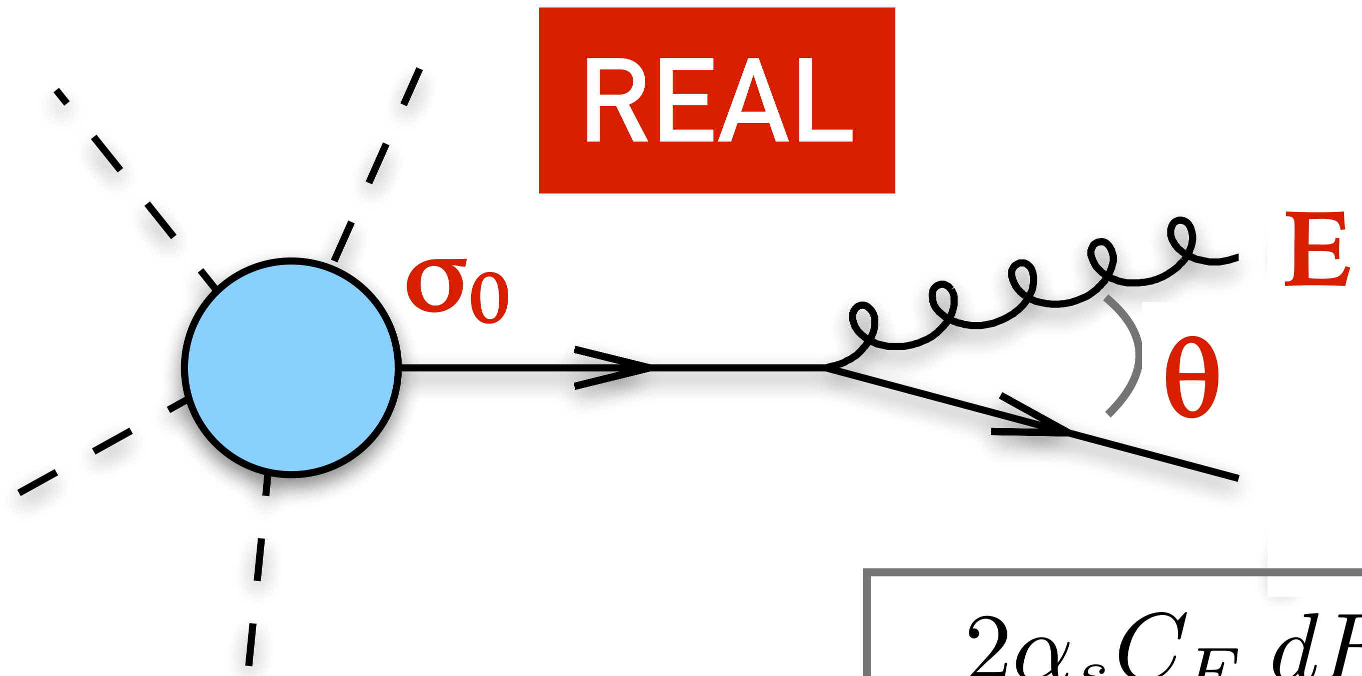


Sterman and Weinberg,
Phys. Rev. Lett. 39, 1436 (1977):

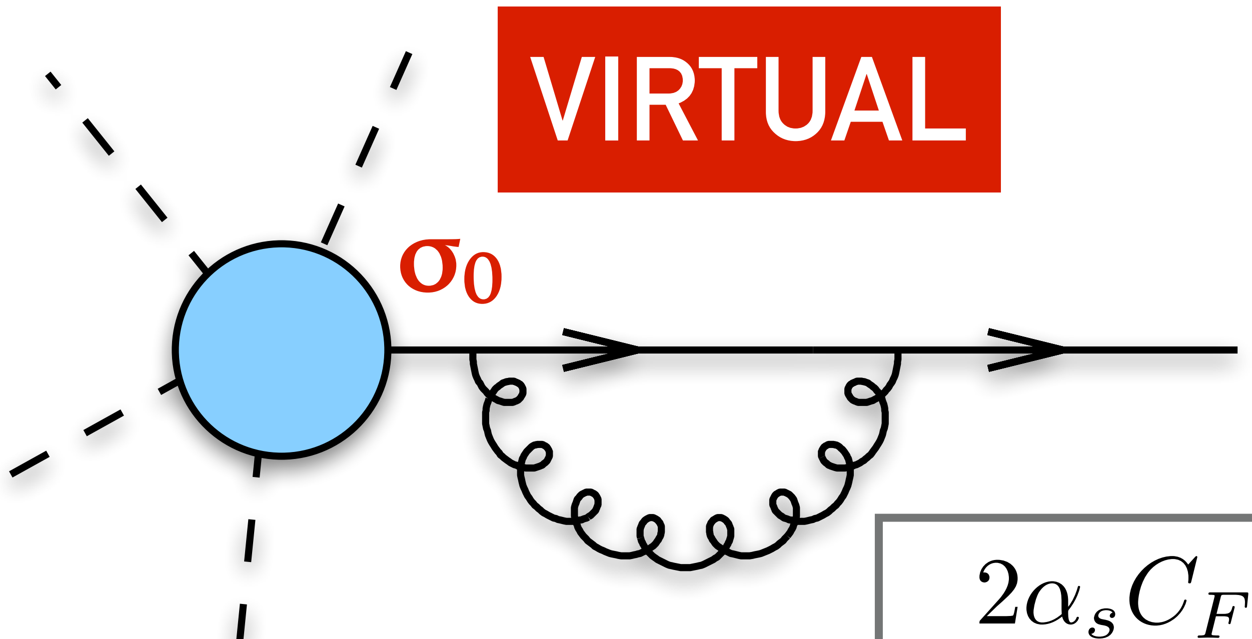
To study jets, we consider the partial cross section $\sigma(E,\theta,\Omega,\varepsilon,\delta)$ for e^+e^- hadron production events, in which all but a fraction $\varepsilon \ll 1$ of the total e^+e^- energy E is emitted within some pair of oppositely directed cones of half-angle $\delta \ll 1$, lying within two fixed cones of solid angle Ω (with $\pi\delta^2 \ll \Omega \ll 1$) at an angle θ to the e^+e^- beam line. We expect this to be measur-

$$\sigma(E,\theta,\Omega,\varepsilon,\delta) = (d\sigma/d\Omega)_0\,\Omega\left[1 - (g_E^2/3\pi^2)\left\{3\ln\delta + 4\ln\delta\,\ln 2\varepsilon + \frac{\pi^3}{3} - \frac{5}{2}\right\}\right]$$

How do we ensure we get finite cross sections in perturbative QCD?



$$+\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$



$$-\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$

Divergences are present in both real and virtual diagrams.

If you are "**infrared and collinear safe**", i.e. your measurement doesn't care whether a soft/collinear gluon has been emitted then the **real and virtual divergences cancel.**

Beyond inclusive cross sections: infrared and collinear (IRC) safety

For an observable's distribution to be calculable in [fixed-order] perturbation theory, the observable should be infra-red safe, i.e. insensitive to the emission of soft or collinear gluons. In particular if $\vec{p}_i$ is any momentum occurring in its definition, it must be invariant under the branching

$$\vec{p}_i \to \vec{p}_j + \vec{p}_k$$

whenever $\vec{p}_j$ and $\vec{p}_k$ are parallel [collinear] or one of them is small [infrared].

[QCD and Collider Physics (Ellis, Stirling & Webber)]

Examples

Multiplicity of gluons is not IRC safe

[modified by soft/collinear splitting]

Energy of hardest particle is not IRC safe

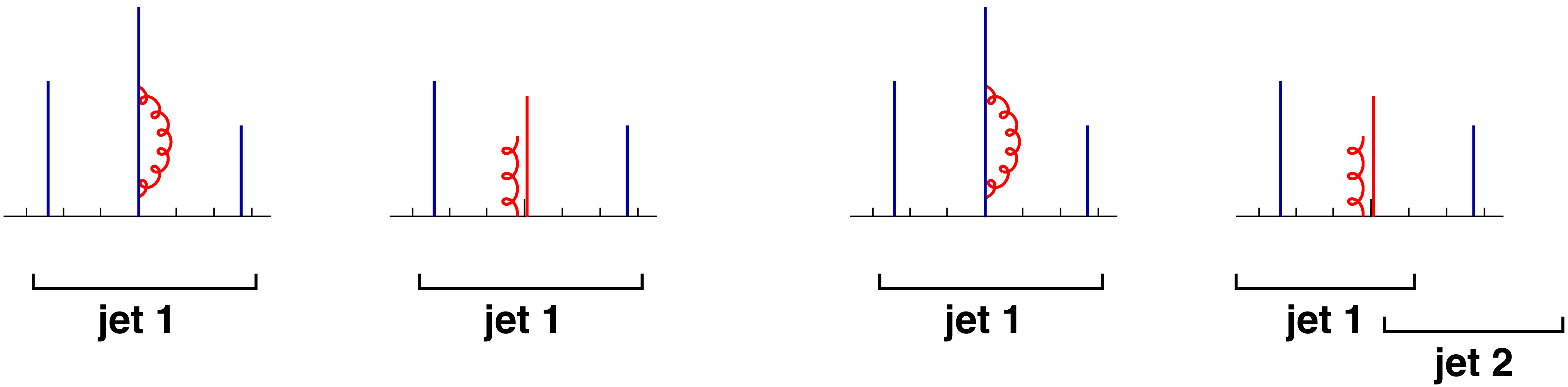
[modified by collinear splitting]

Energy flow into a cone is IRC safe

[soft emissions don't change energy flow,
collinear emissions don't change its direction]

Key requirement: infrared and collinear safety

Collinear Safe | **Collinear Unsafe**



α_s^n x $(-\infty)$ α_s^n x $(+\infty)$ α_s^n x $(-\infty)$ α_s^n x $(+\infty)$

Infinities cancel | **Infinities do not cancel**

Invalidates perturbation theory

hadron collider jet algorithms

Two parameters, *R* (jet opening angle) **and *$p_{t,min}$***
(These are the two parameters in essentially every widely used hadron-collider jet algorithm)

Sequential recombination algorithm

$$d_{ij} = \min(p_{ti}^2, p_{tj}^2)\frac{\Delta R_{ij}^2}{R^2}, \qquad d_{iB} = p_{ti}^2, \qquad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

Define d_{ij} distance between every pair of particles as squared transverse momentum (k_t) of softer of *i* and *j* relative to harder one.

It will be small when particles are collinear ($\Delta R_{ij} \ll 1$) or if one of the particles is soft (p_{ti} or $p_{tj} \ll$ hard scale).

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1. Find smallest of d_{ij}, d_{iB}
2. If *ij*, recombine particles i and j
3. If *iB*, call i a jet and remove from list of particles
4. repeat from step 1 until no particles left

   Only use jets with $p_t > p_{t,min}$

Inclusive k_t algorithm
S.D. Ellis & Soper, 1993
Catani, Dokshitzer, Seymour & Webber, 1993

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1. Find smallest of d_{ij}, d_{iB}
2. If *ij*, recombine particles i and j
3. If *iB*, call *i* a jet and remove from list of particles

If a particle i has no neighbours j within a distance $\Delta R_{ij} \leq R$, then $d_{iB} <$ all d_{ij}, and i becomes a jet.

k_t in action

$\mathbf{k_t}$ alg.: Find smallest of

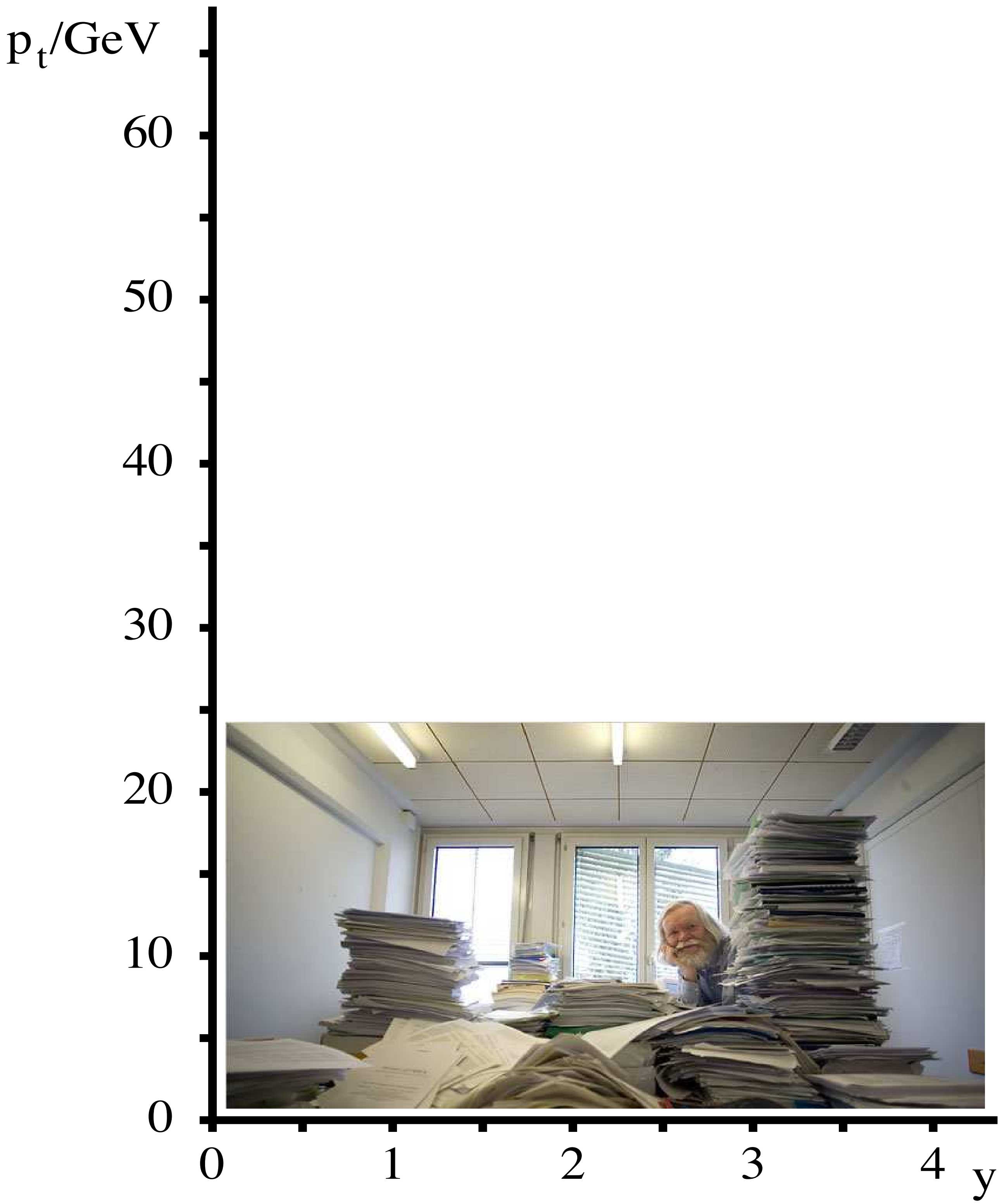
$$d_{ij} = \min(k_{ti}^2, k_{tj}^2)\frac{\Delta R_{ij}^2}{R^2}, \quad d_{iB} = k_{ti}^2$$

- If d_{ij} recombine
- if d_{iB}, i is a jet

Example clustering with k_t algorithm, $R = 1.0$

ϕ assumed 0 for all towers

k_t in action



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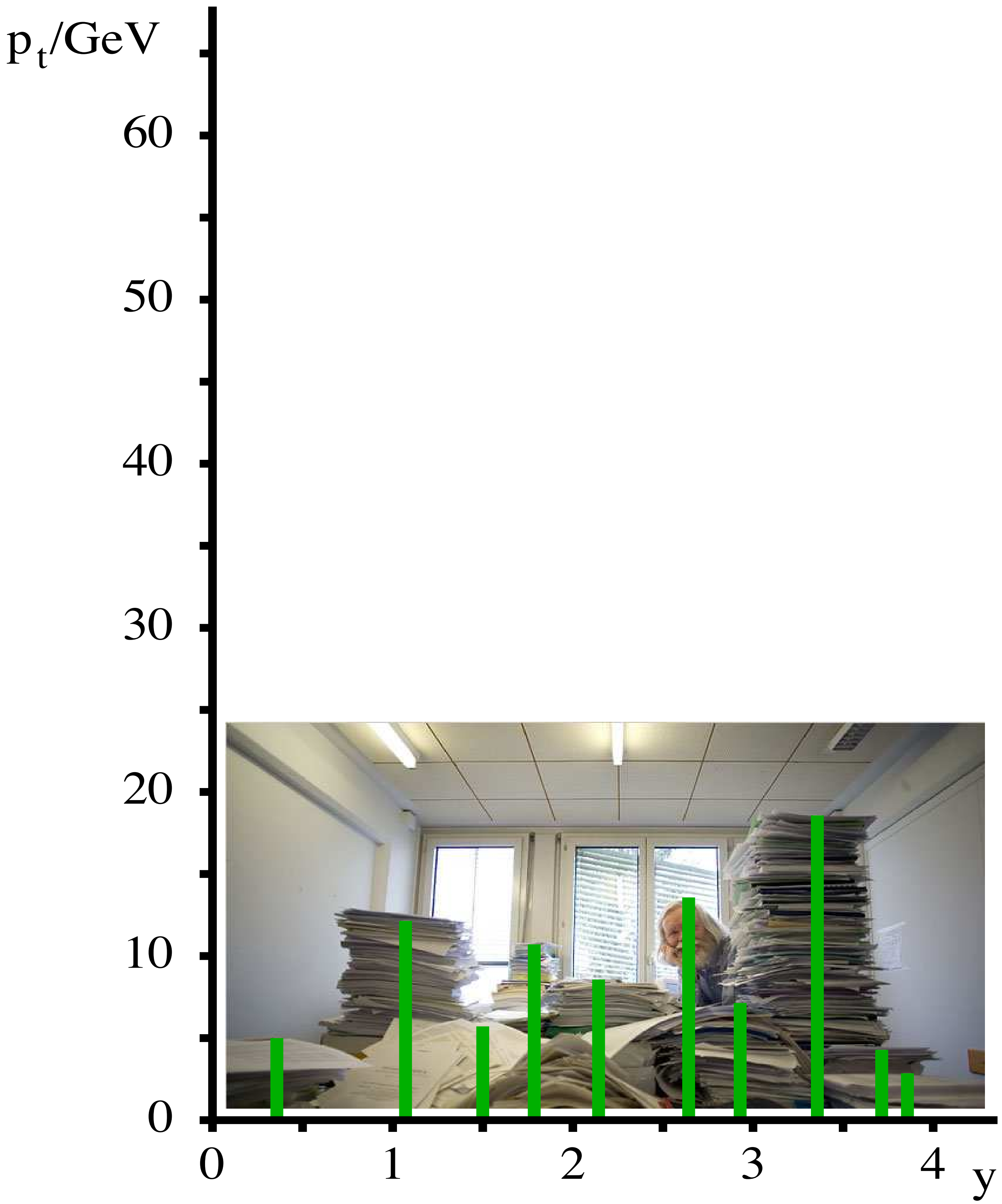

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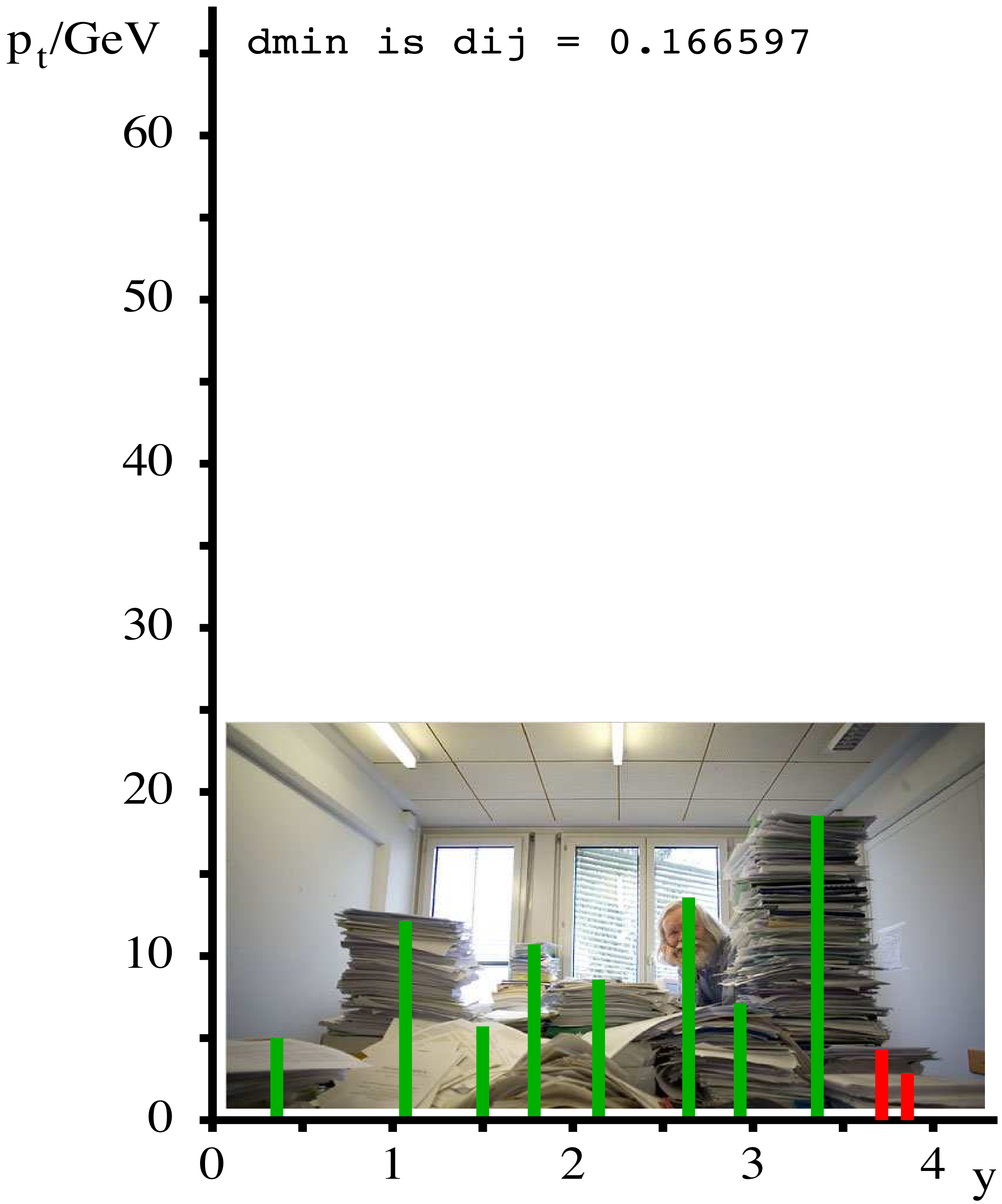

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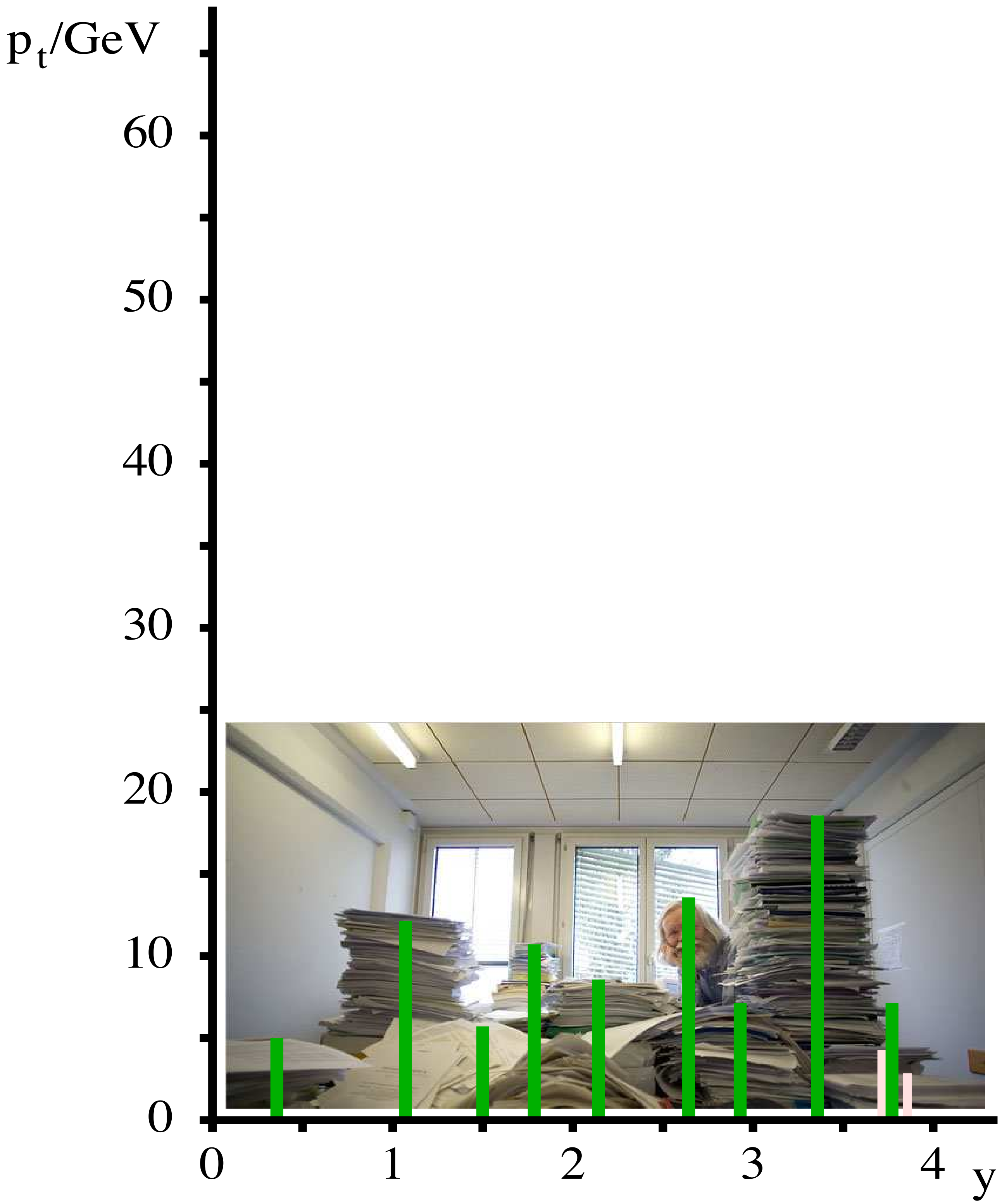

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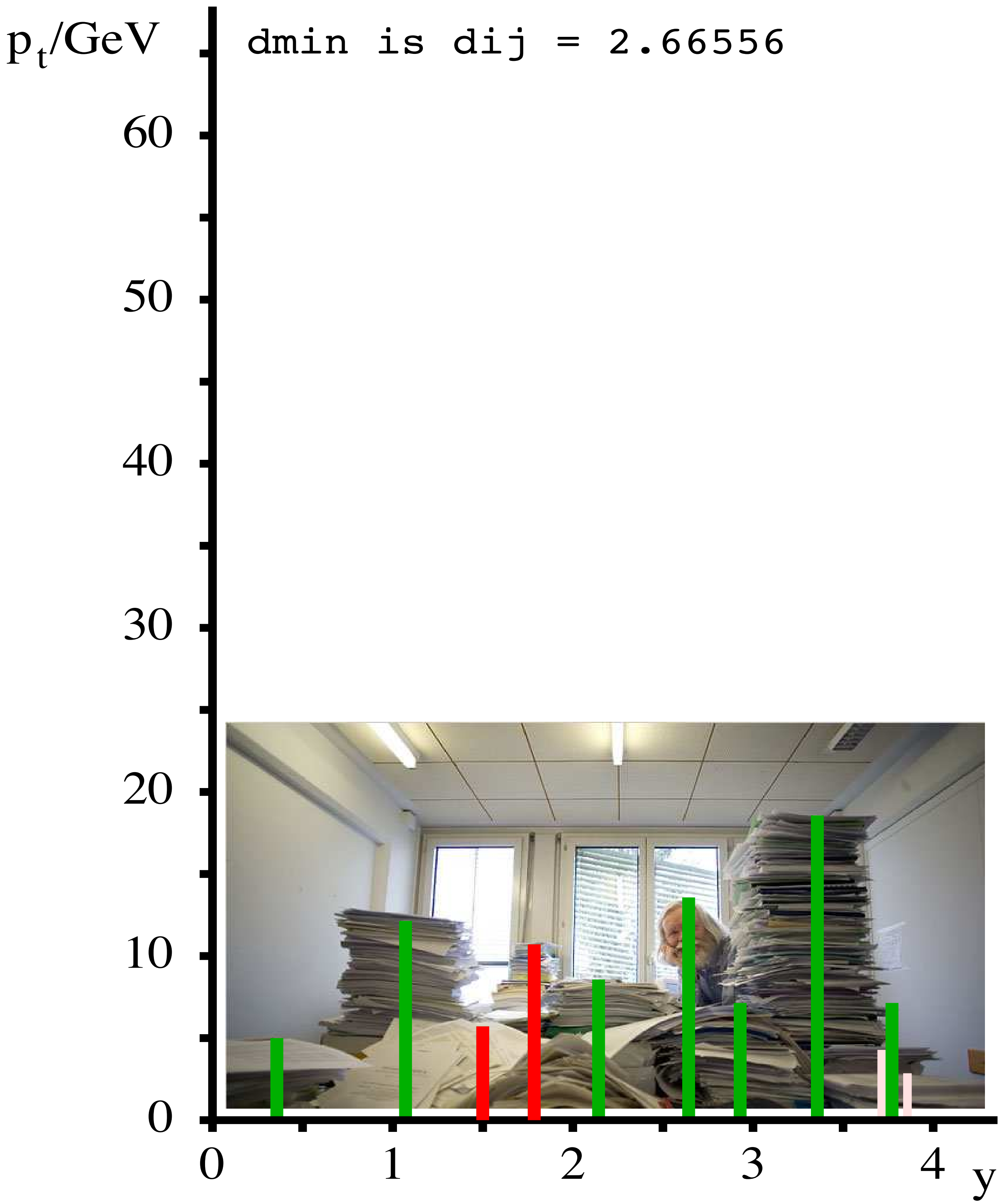

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- if d_{iB}, i is a jet

Example clustering with k_t algorithm, $R = 1.0$

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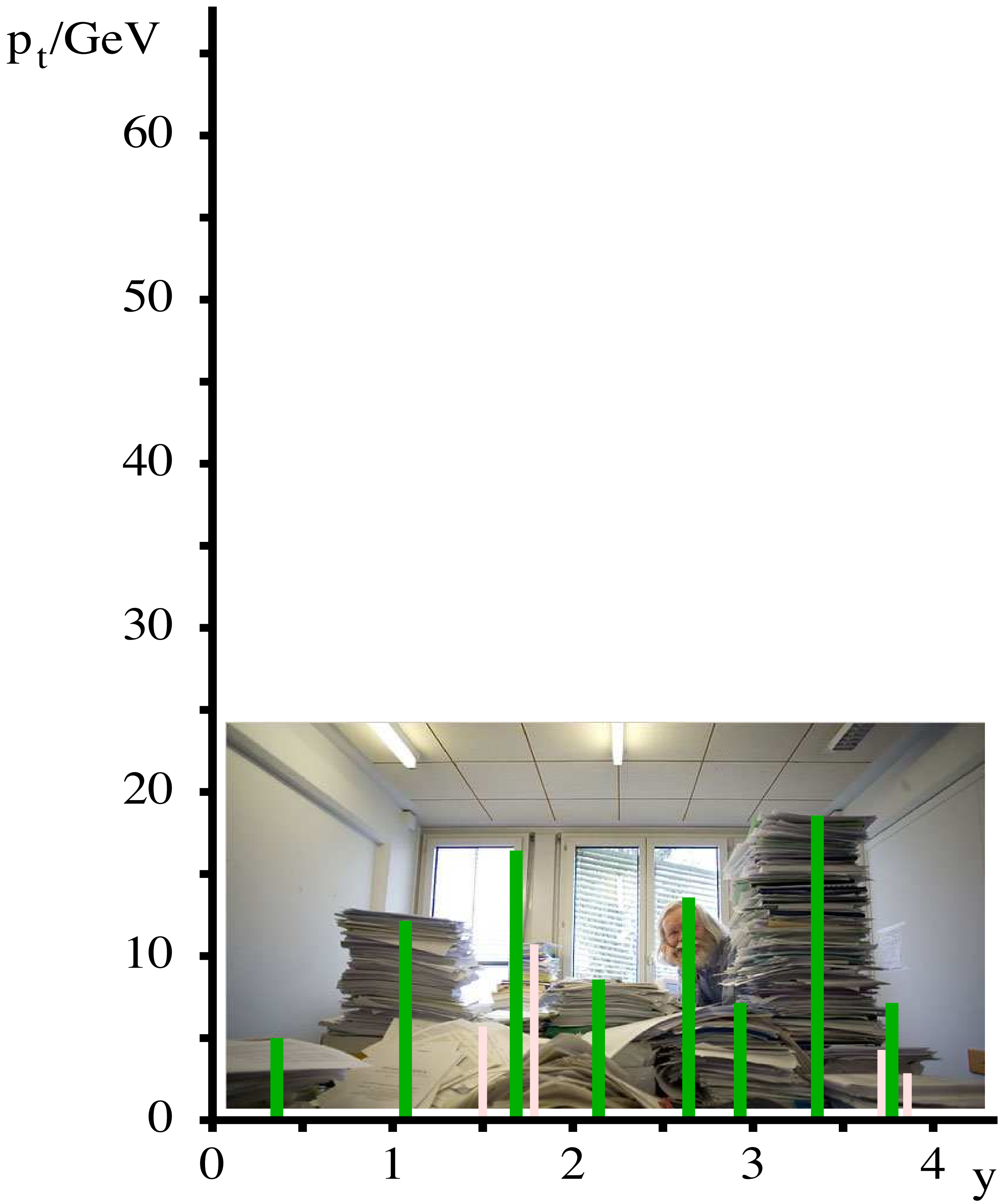

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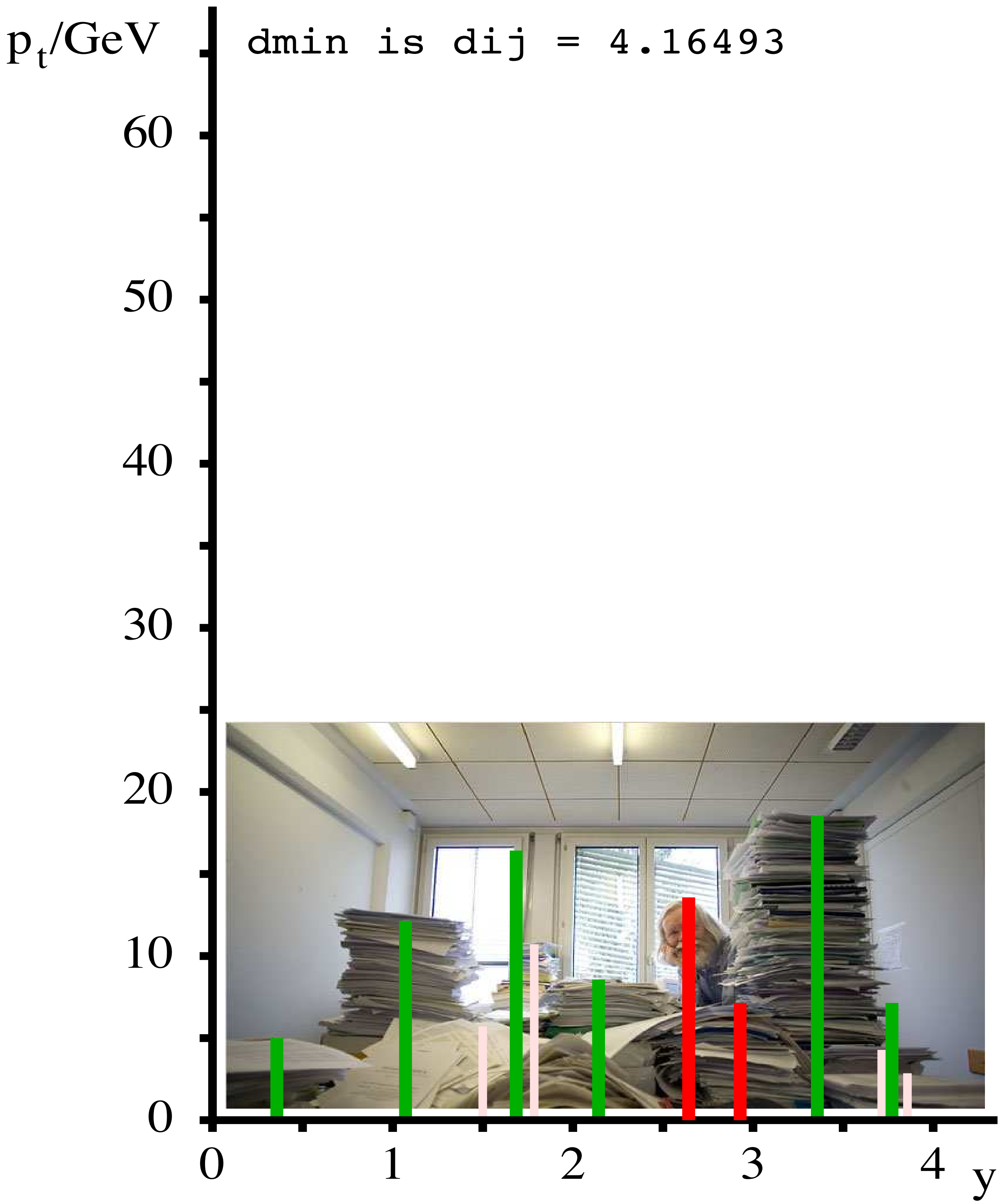

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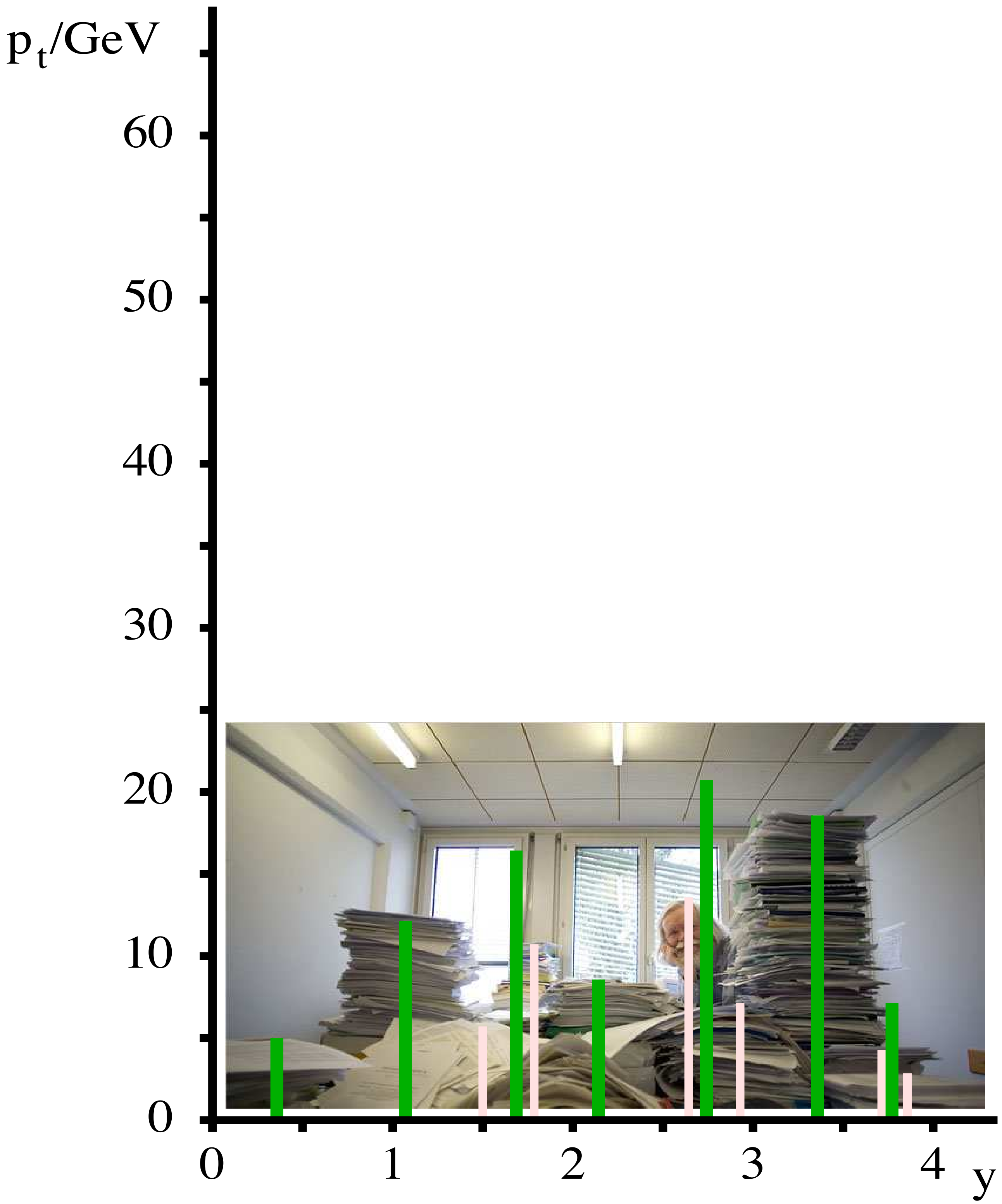

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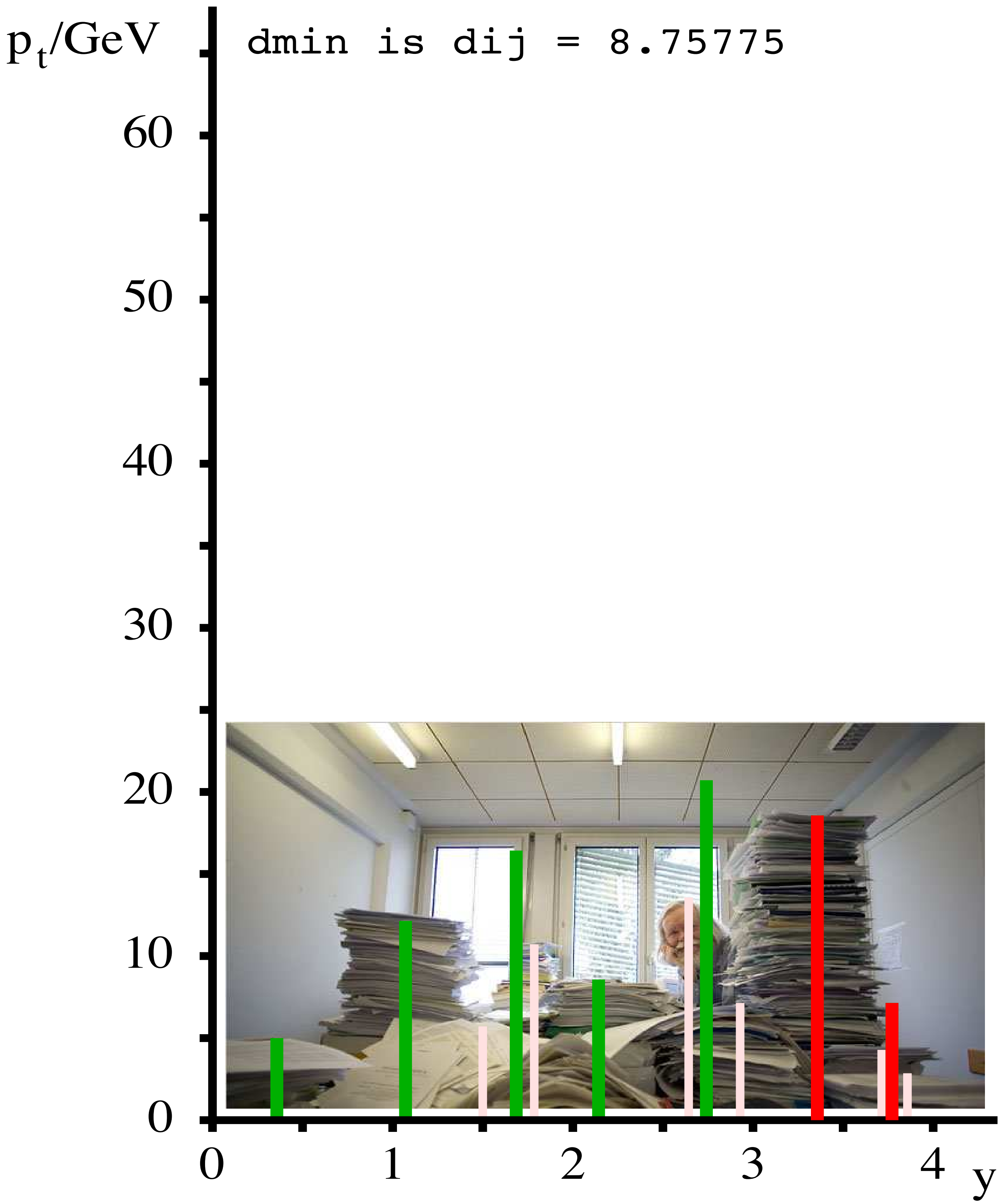

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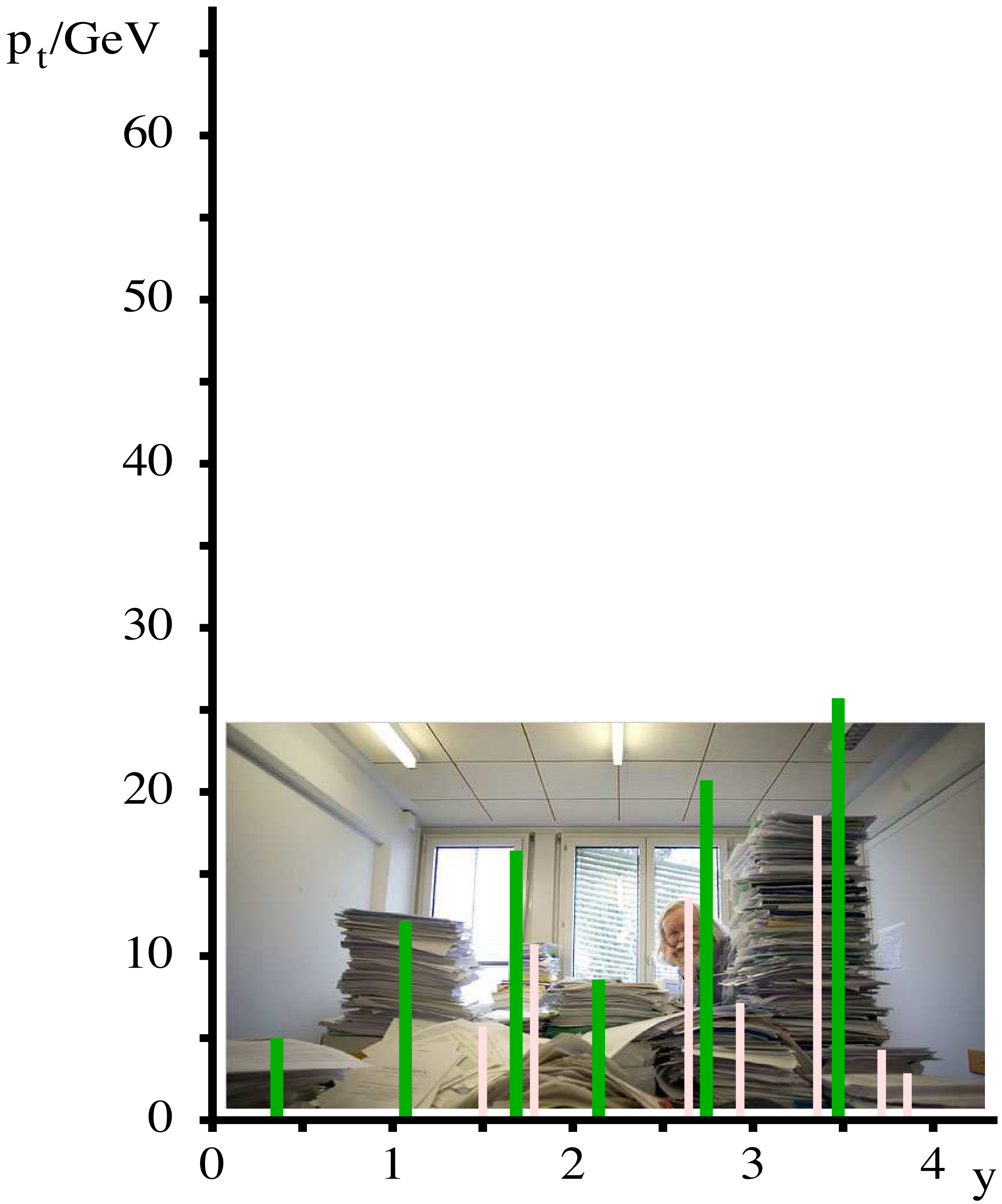

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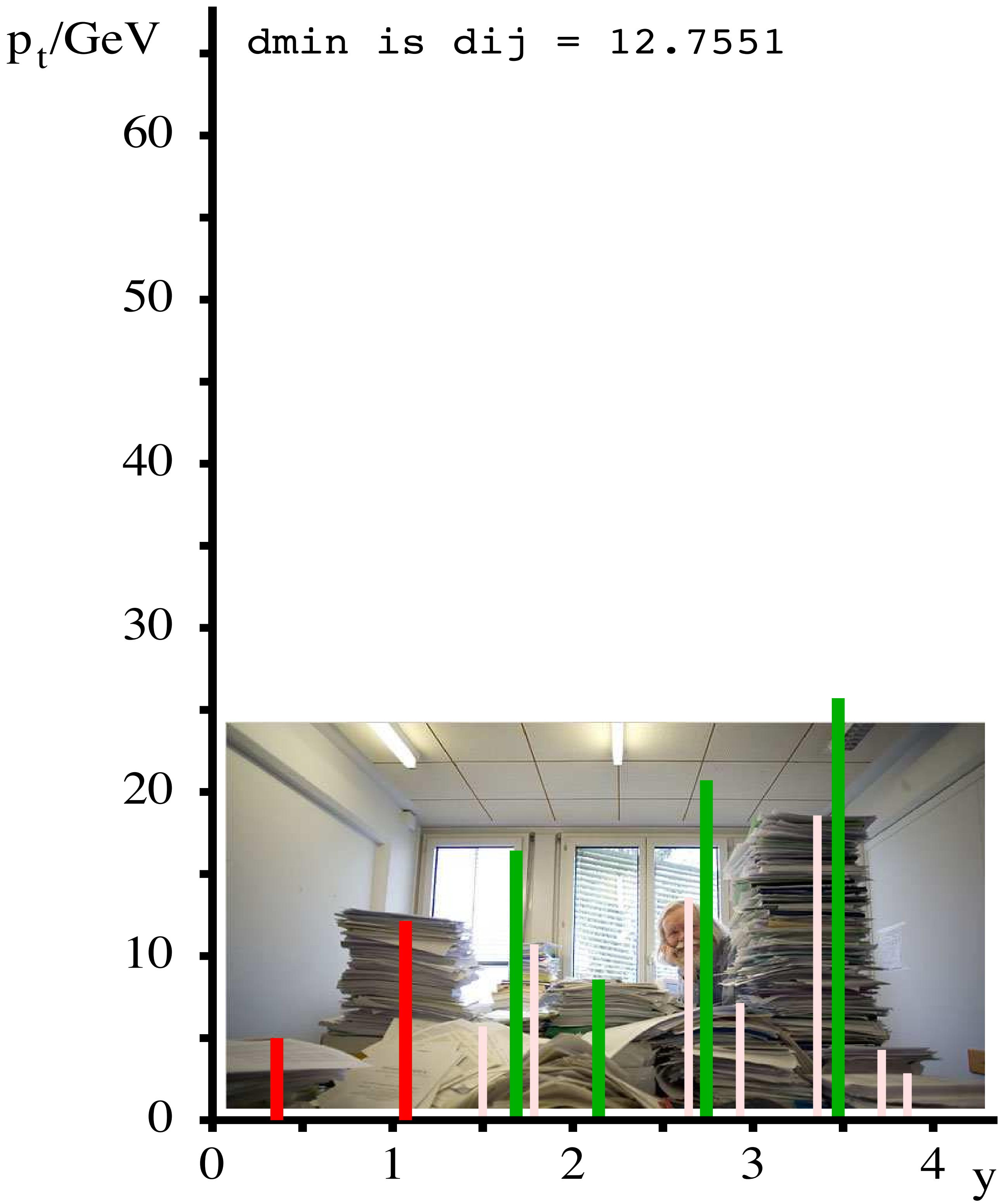

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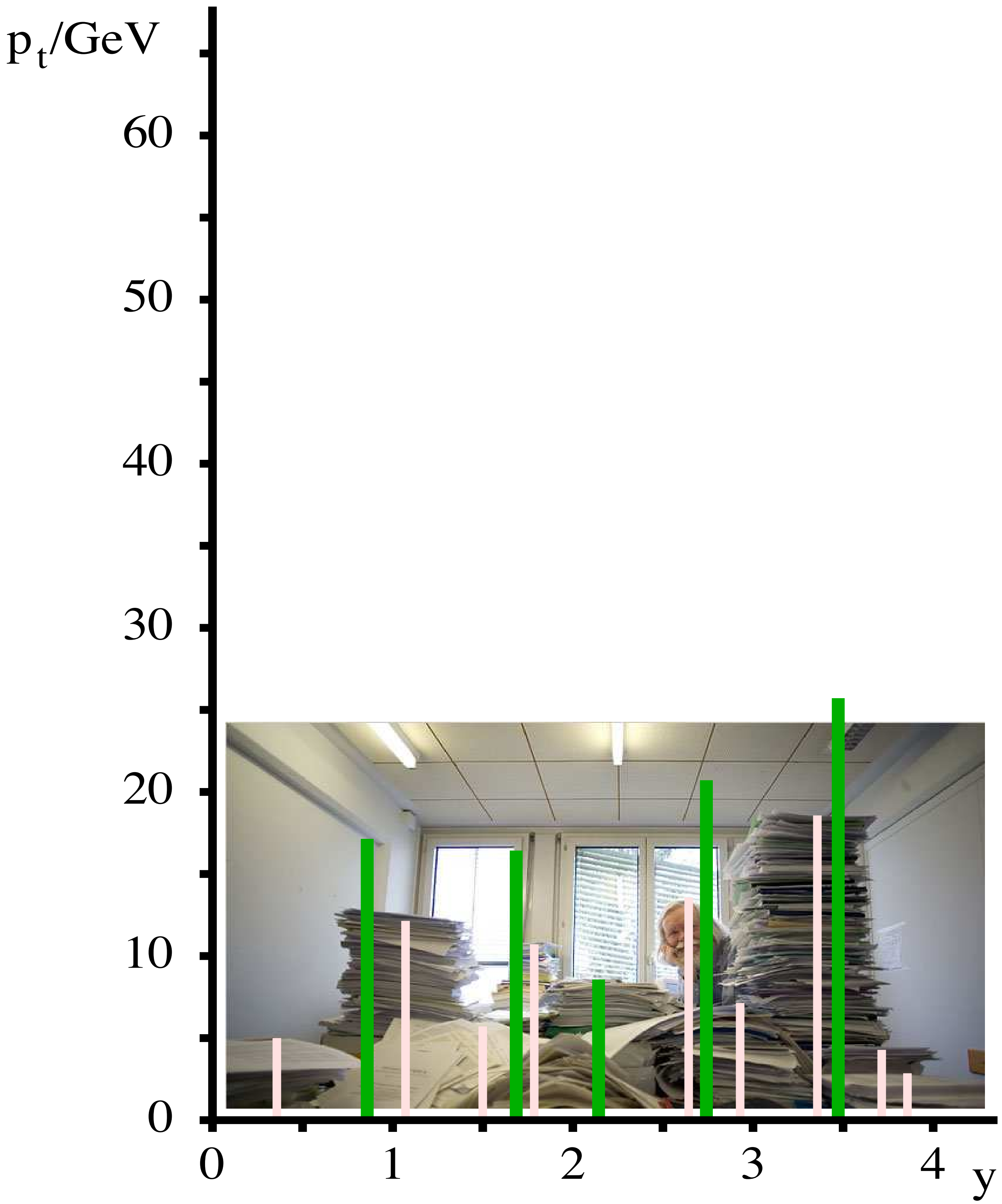

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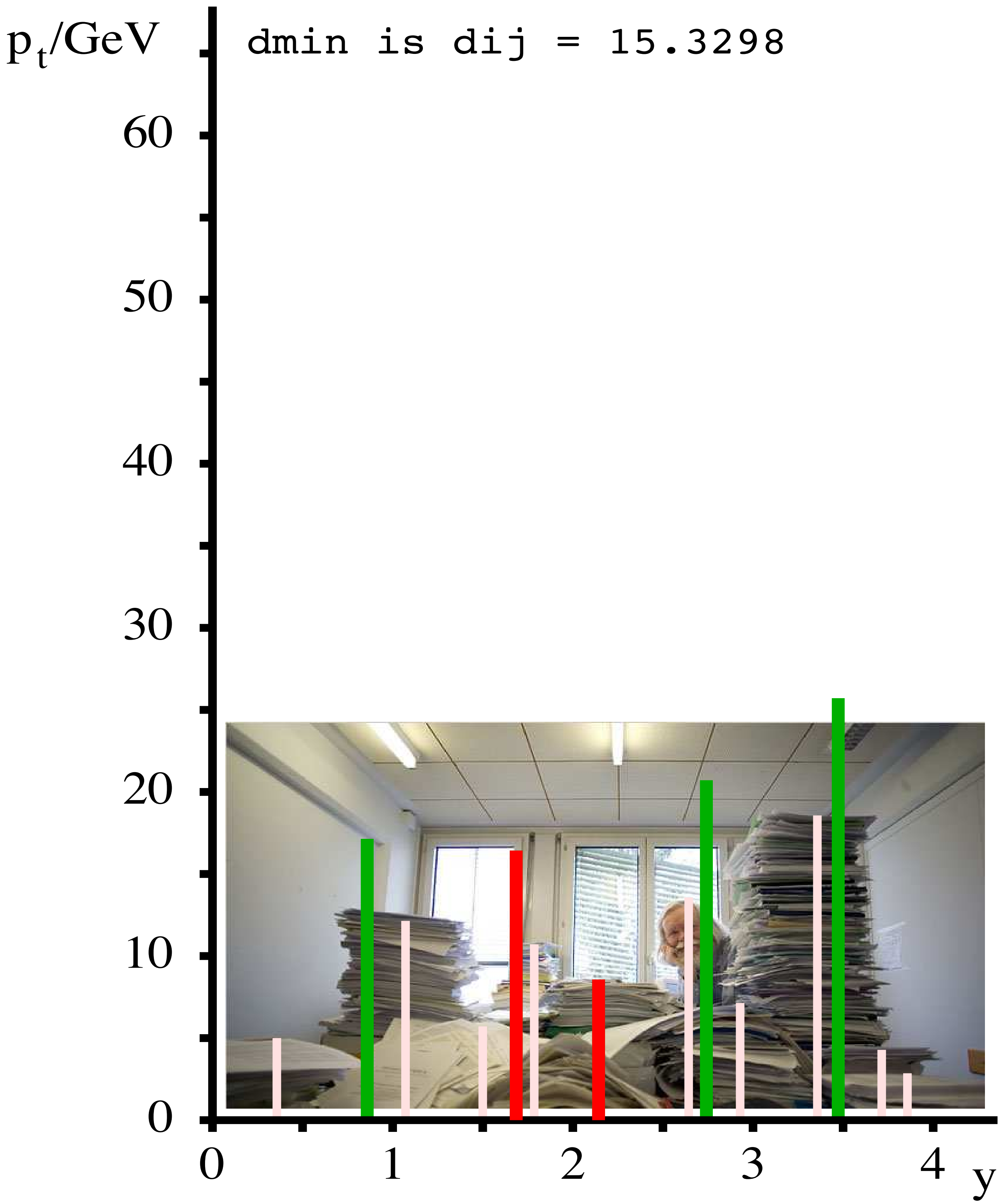

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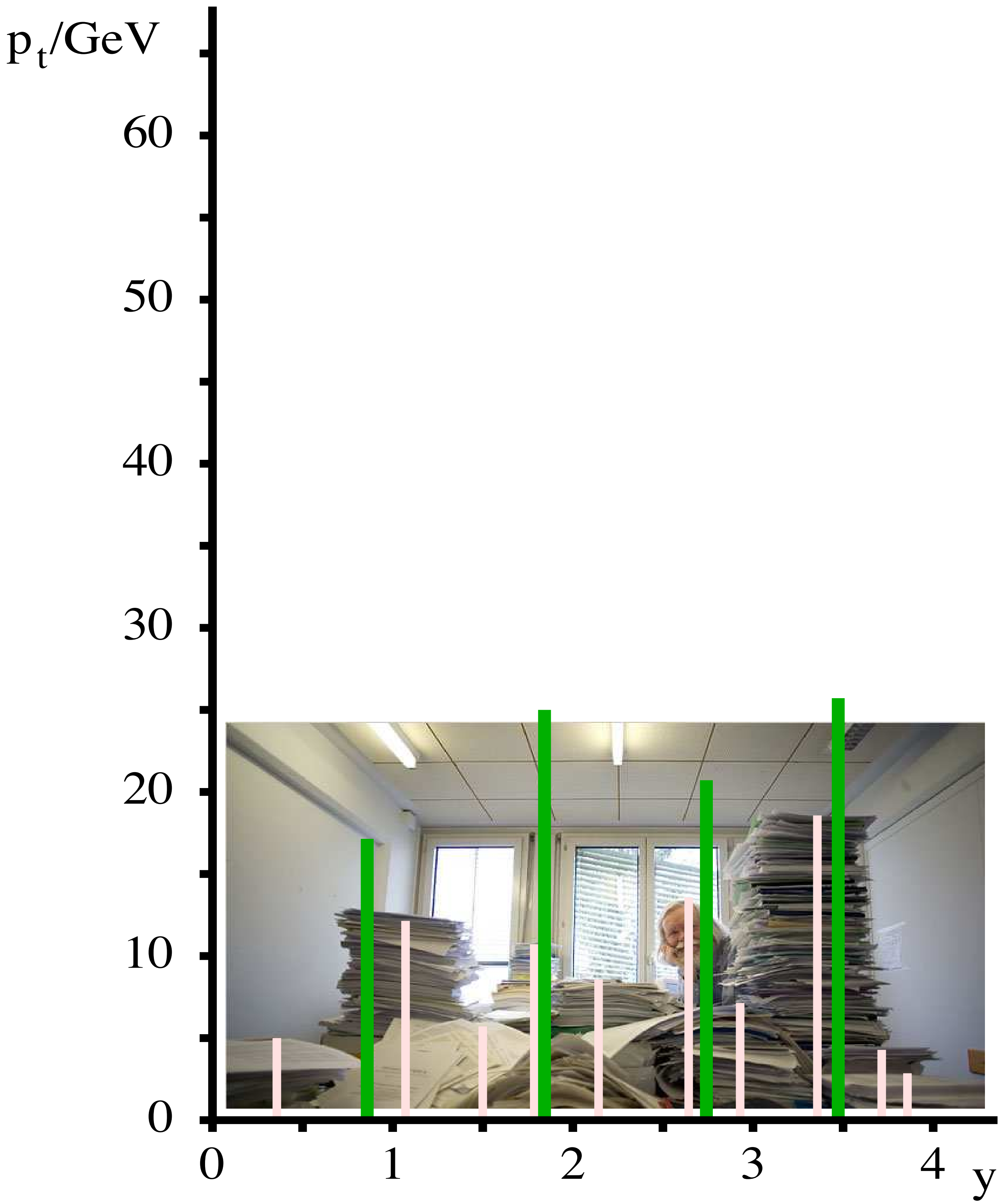

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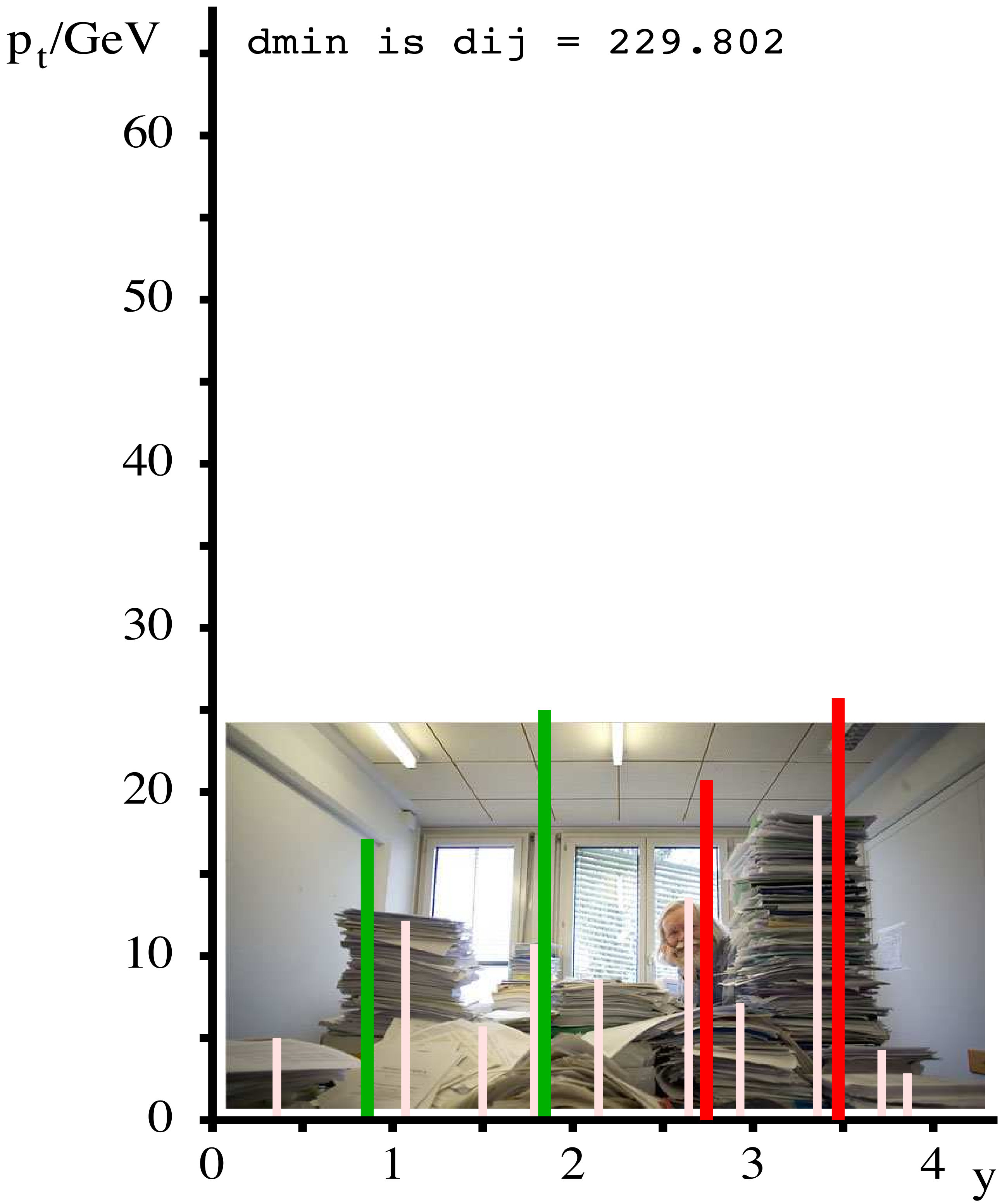

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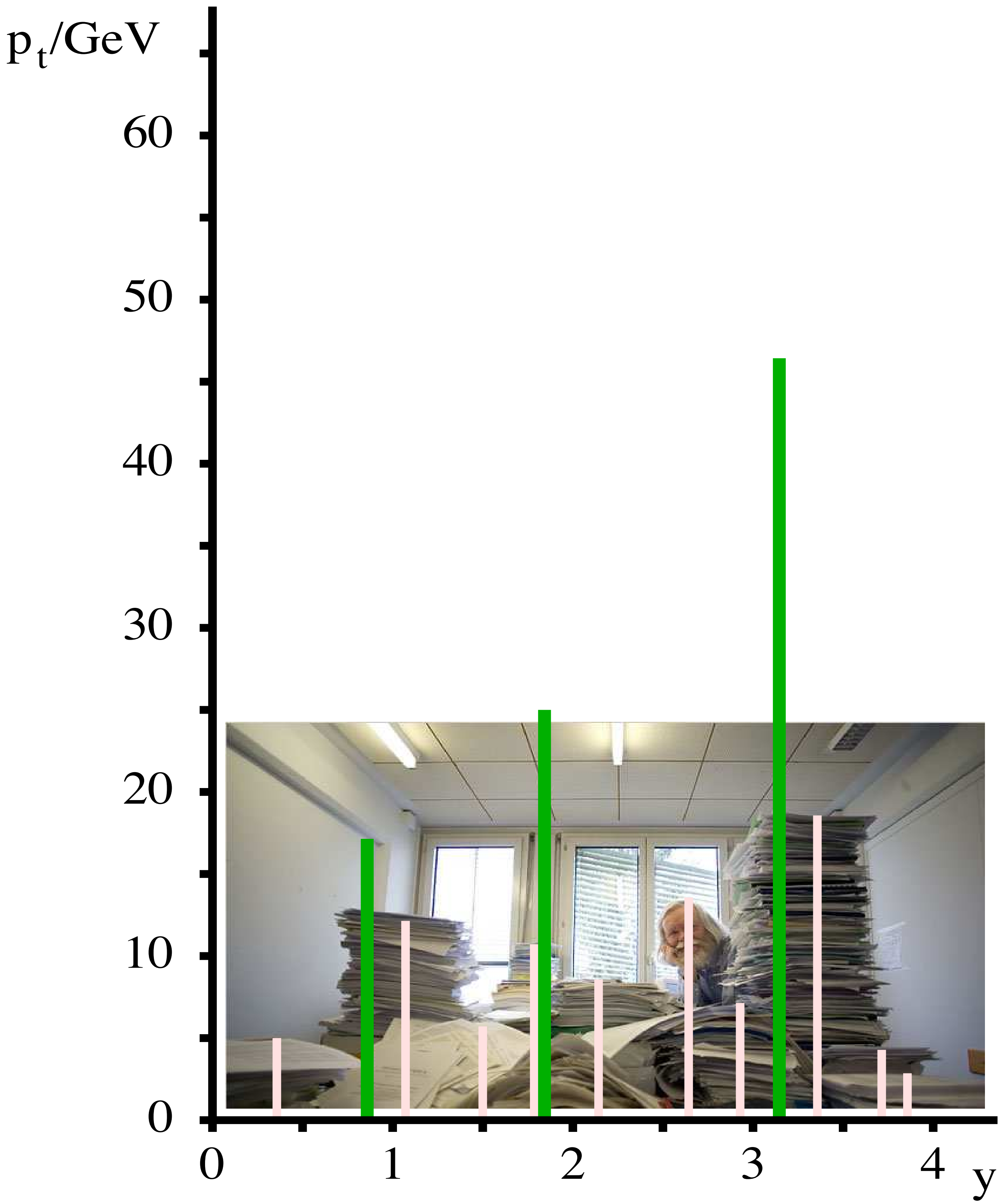

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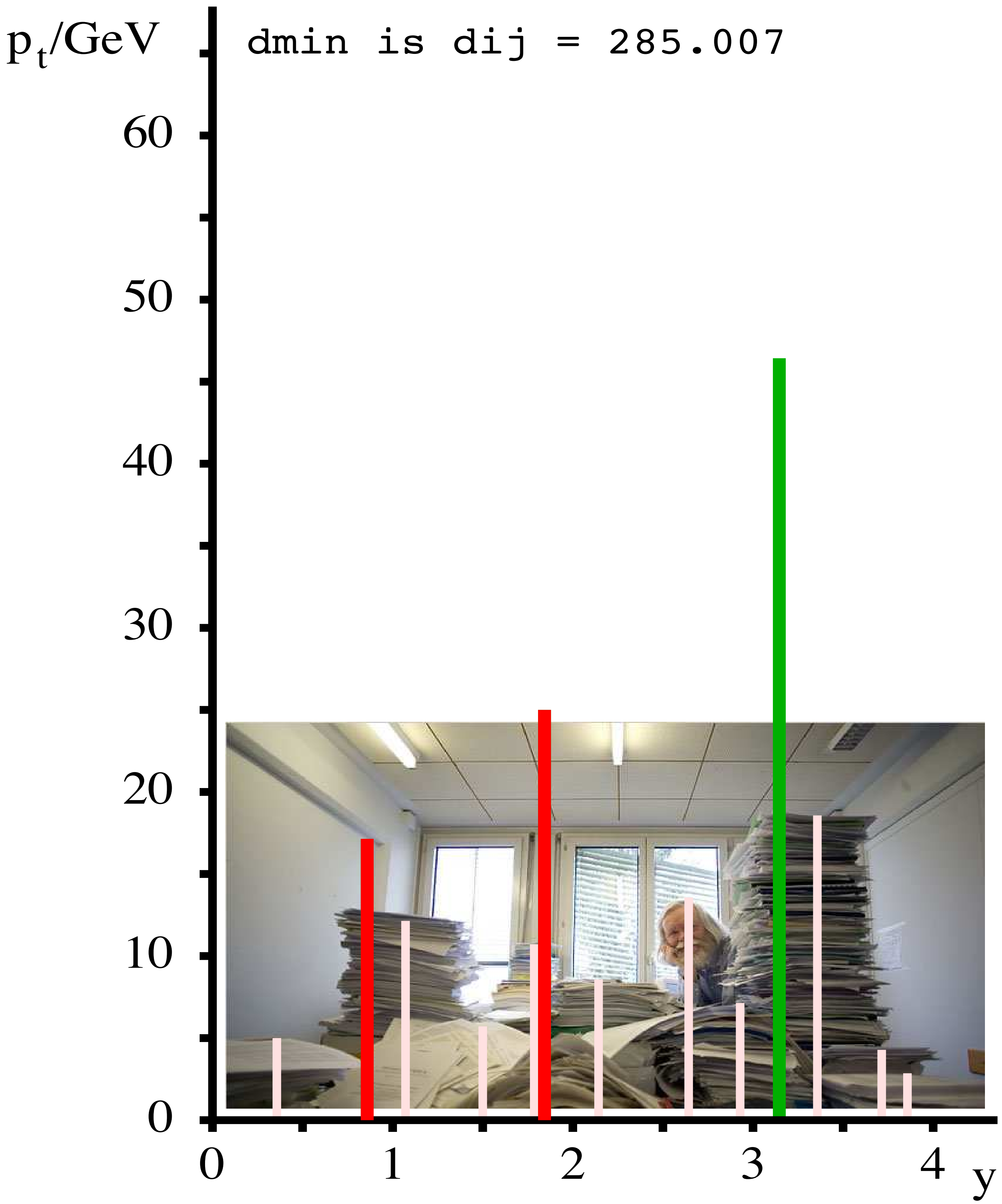

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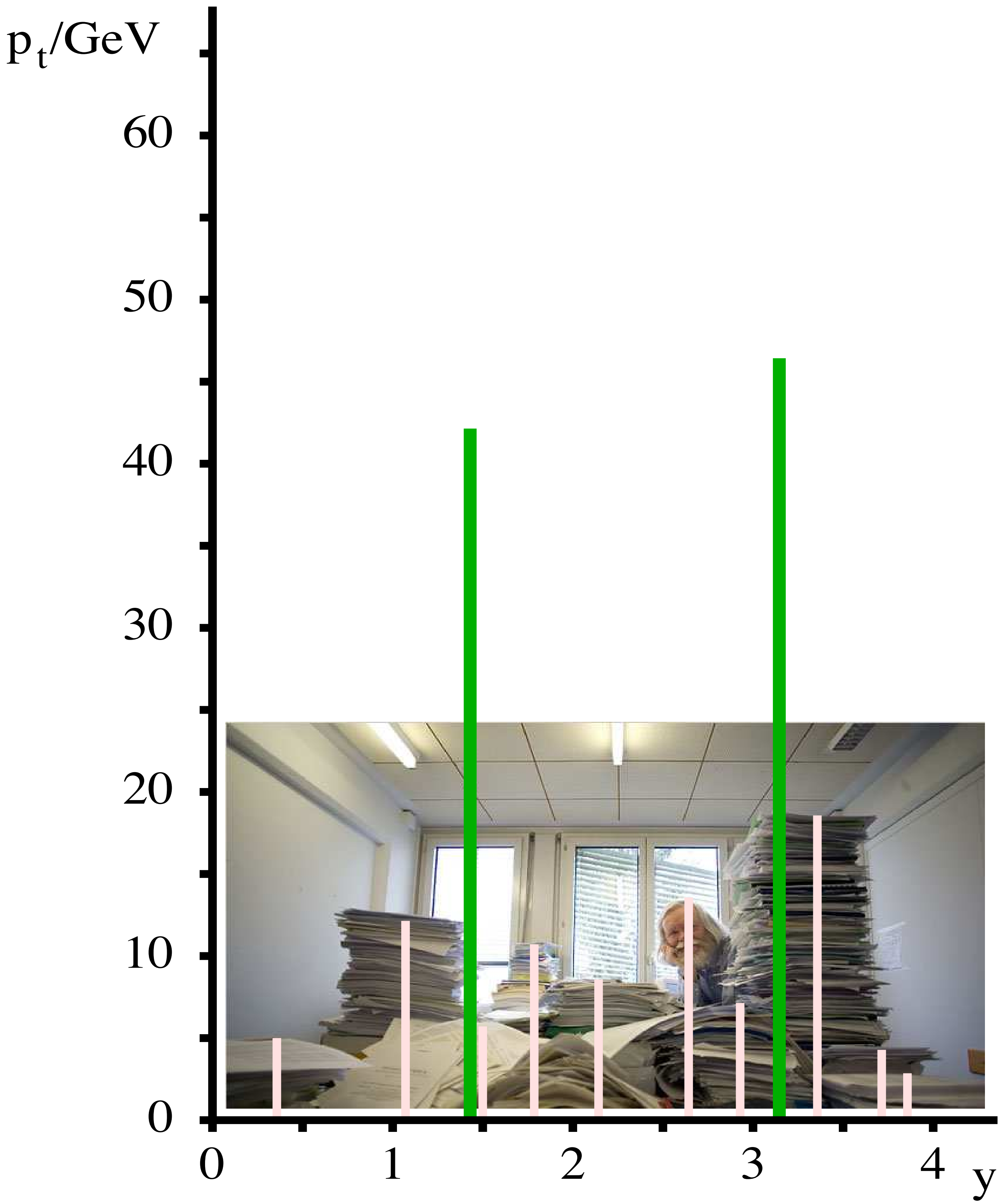

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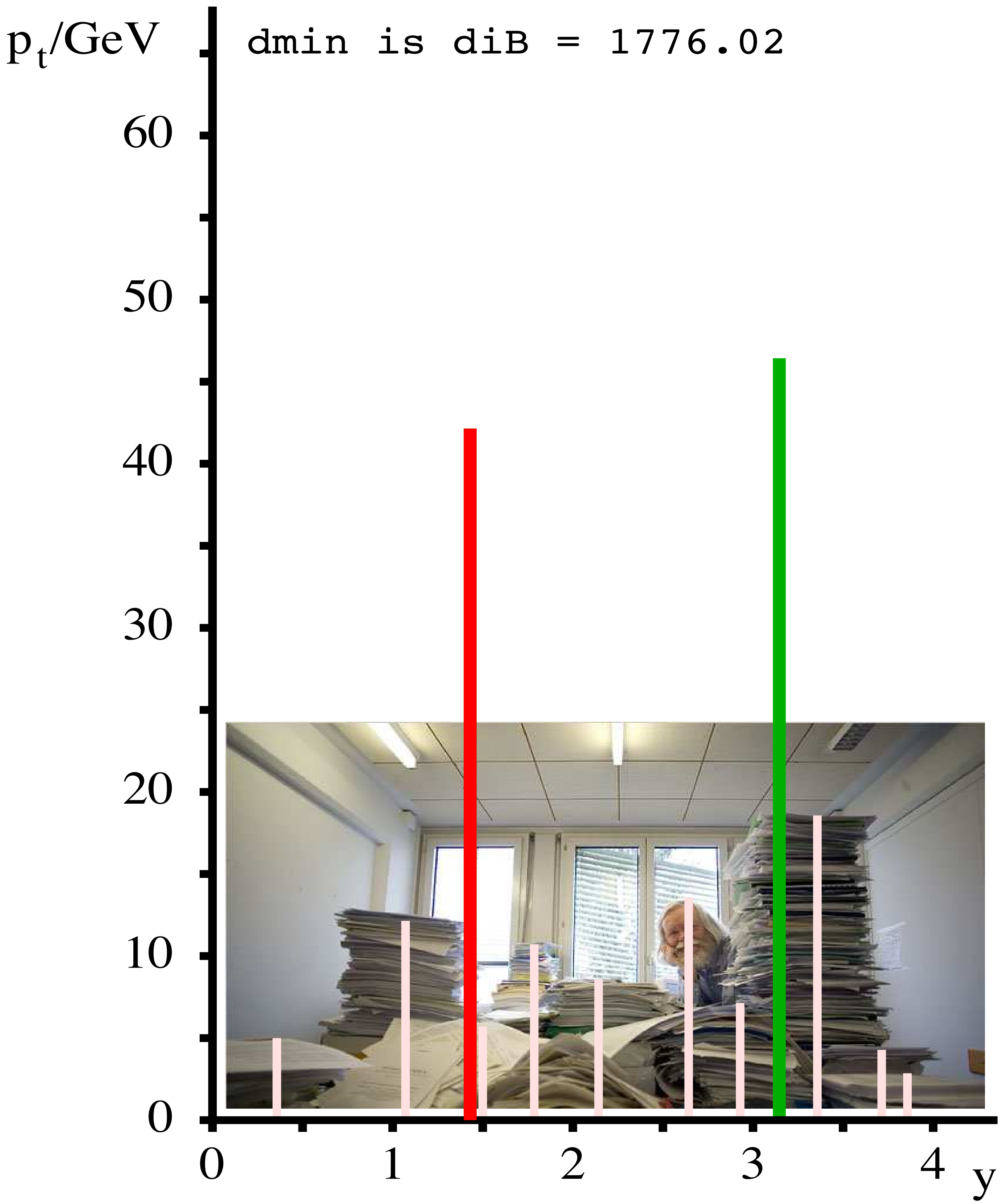

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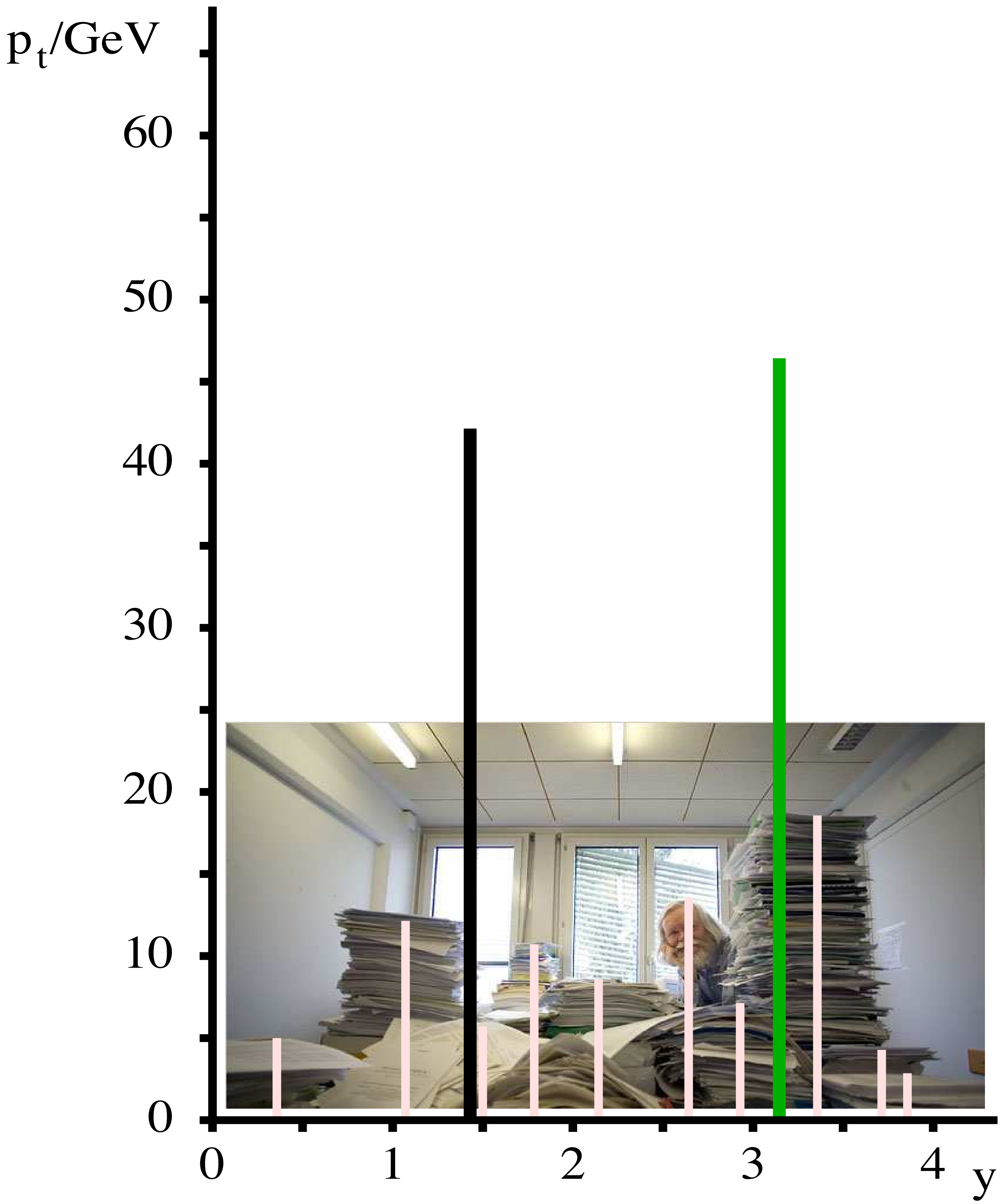

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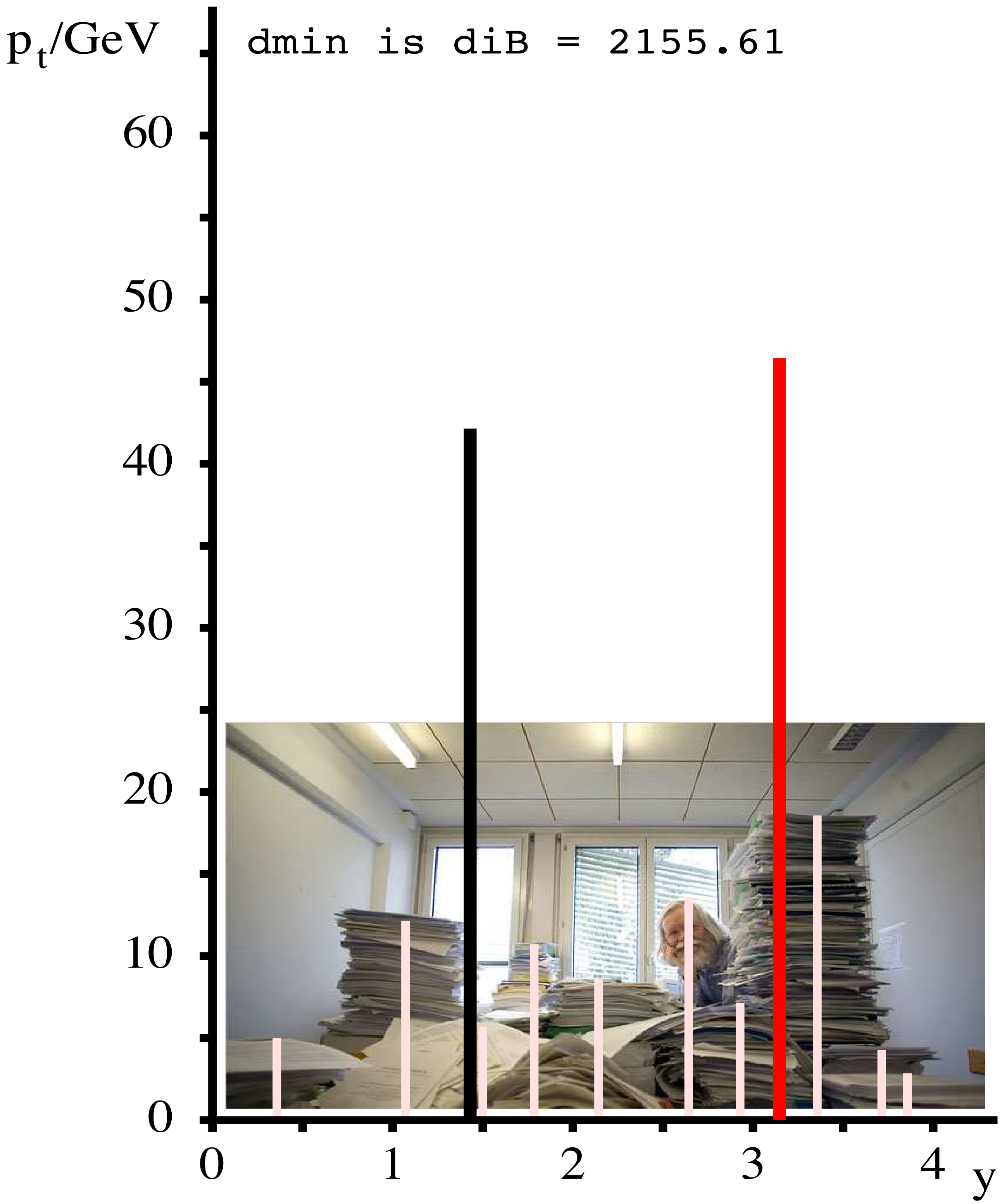

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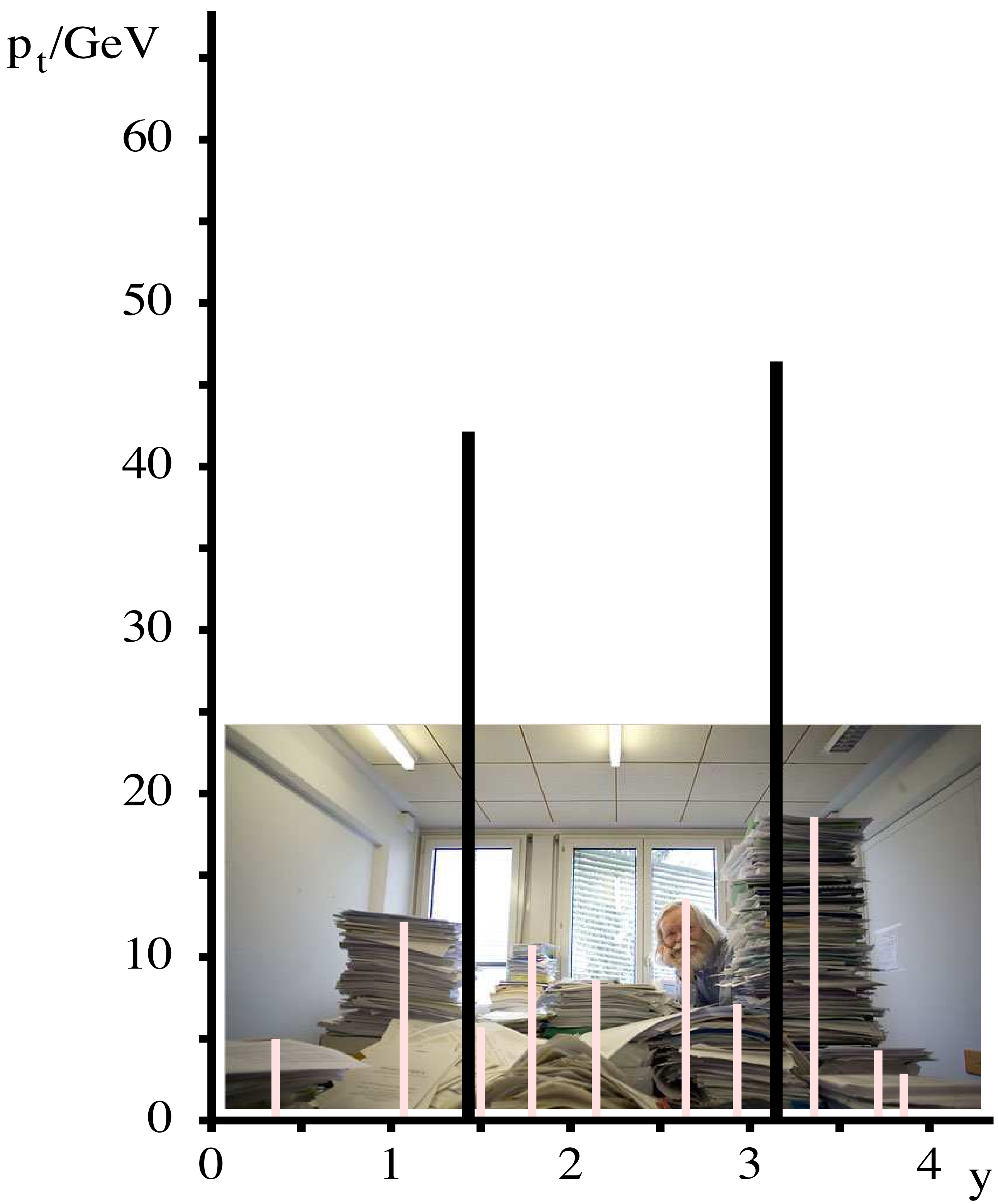

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Example clustering with k_t algorithm, $R = 1.0$

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Cambridge/Aachen: the simplest of hadron-collider algorithms

- Recombine pair of objects closest in ΔR_{ij}
- Repeat until all $\Delta R_{ij} > R$ — remaining objects are jets

Dokshitzer, Leder, Moretti, Webber '97 (Cambridge): more involved e^+e^- form
Wobisch & Wengler '99 (Aachen): simple inclusive hadron-collider form
One still applies a $p_{t,\min}$ cut to the jets, as for inclusive k_t

C/A privileges the collinear divergence of QCD;
it 'ignores' the soft one

Anti-k_t: formulated similarly to inclusive k_t, but with

$$d_{ij} = \frac{1}{\max(p_{ti}^2, p_{tj}^2)} \frac{\Delta R_{ij}^2}{R^2}, \qquad d_{iB} = \frac{1}{p_{ti}^2}$$

Cacciari, GPS & Soyez '08 [+Delsart unpublished]

Anti-k_t privileges the collinear divergence of QCD and disfavours clustering between pairs of soft particles

Most pairwise clusterings involve at least one hard particle

Clustering grows around hard cores

$$d_{ij} = \frac{1}{\max(p_{ti}^2, p_{tj}^2)} \frac{\Delta R_{ij}^2}{R^2}, \qquad d_{iB} = \frac{1}{p_{ti}^2}$$

Anti-k_t in action

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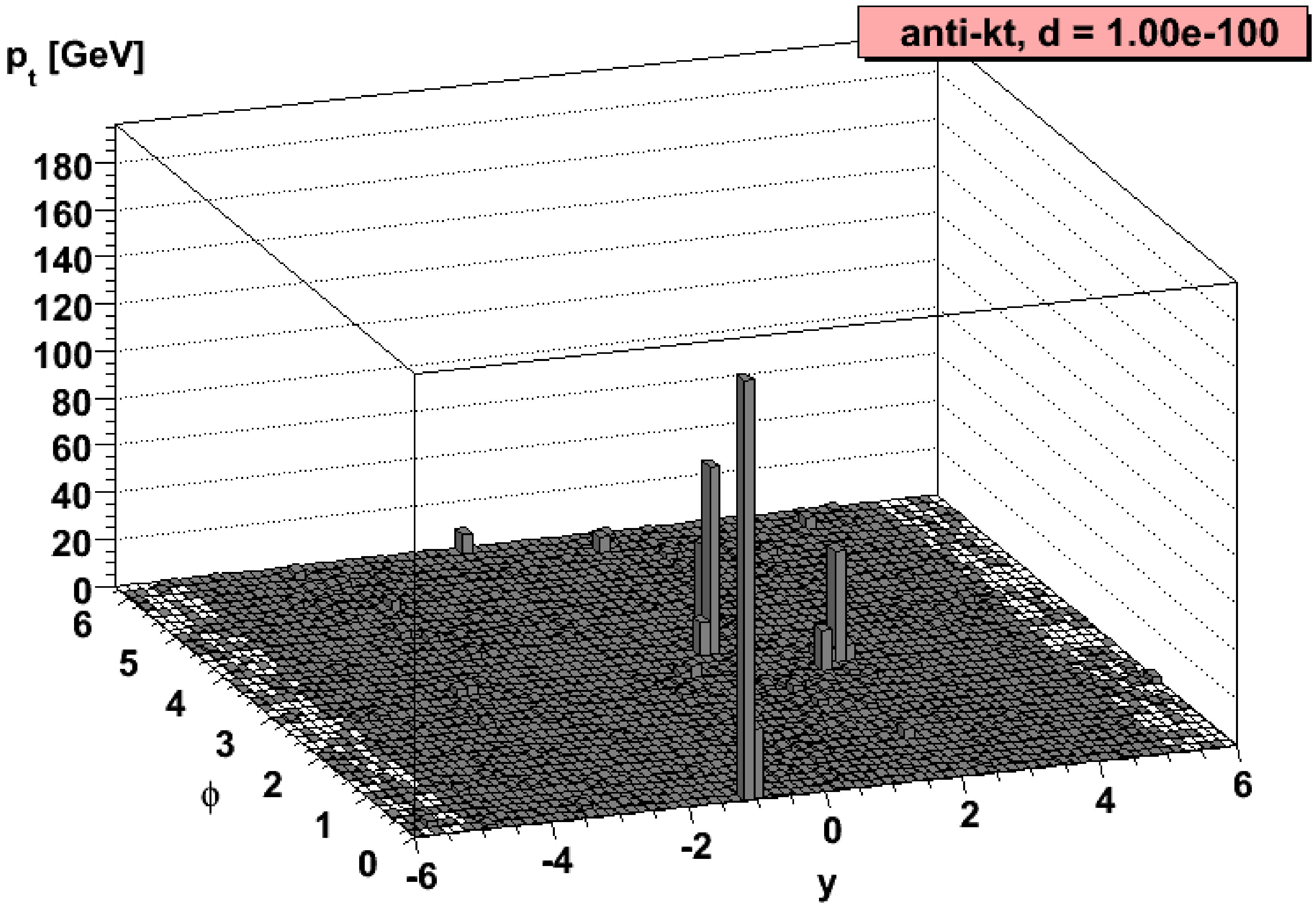

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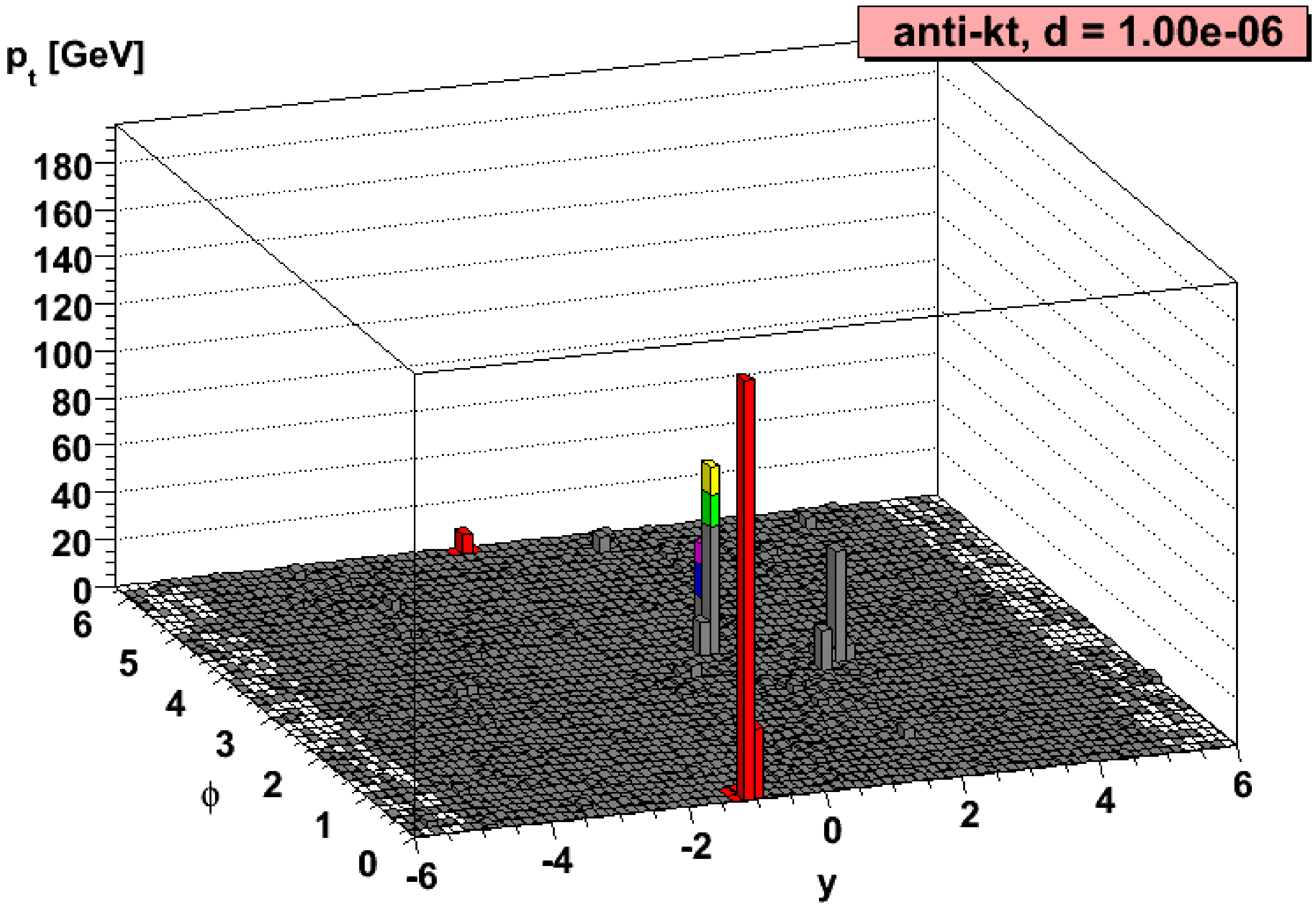

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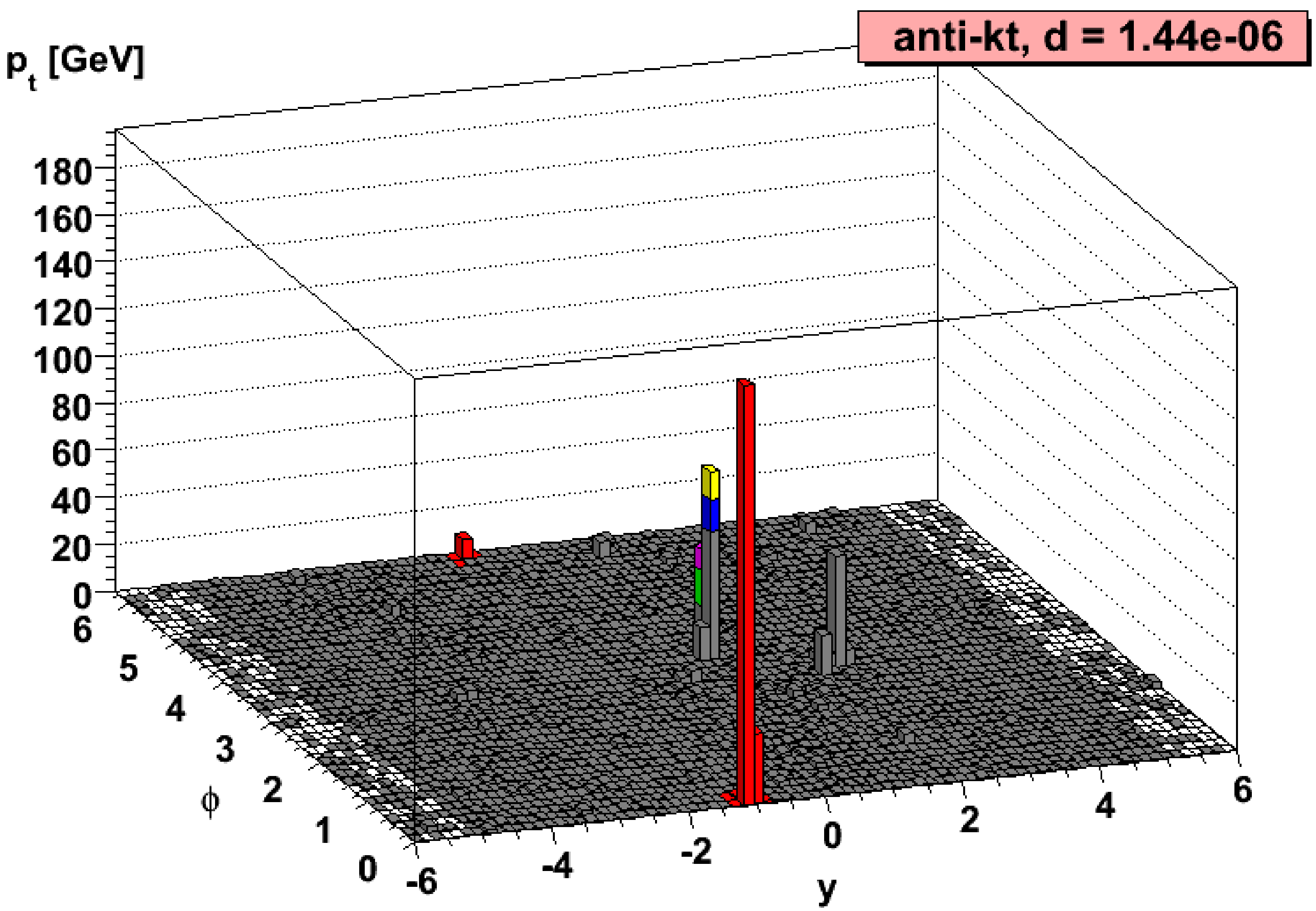

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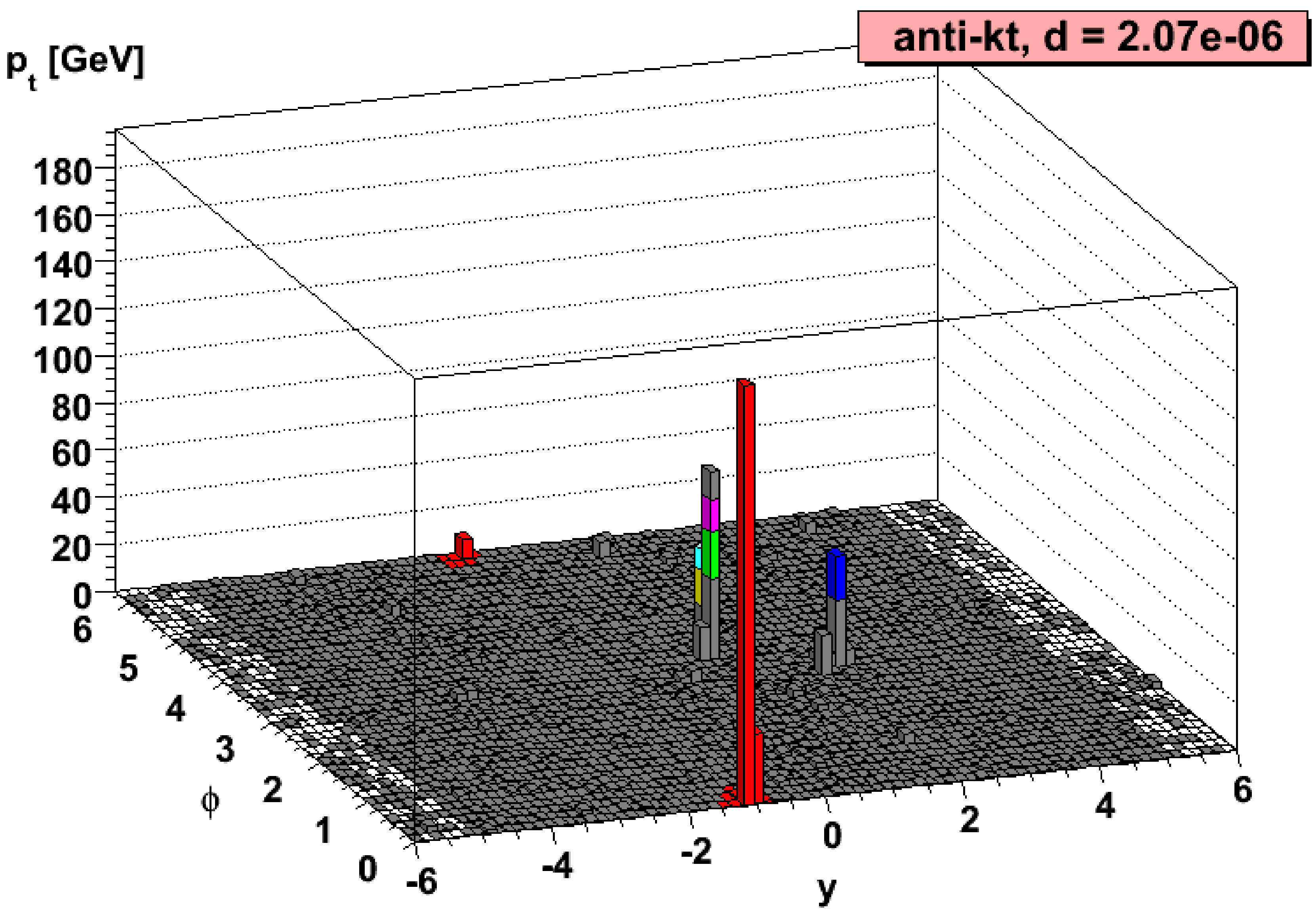

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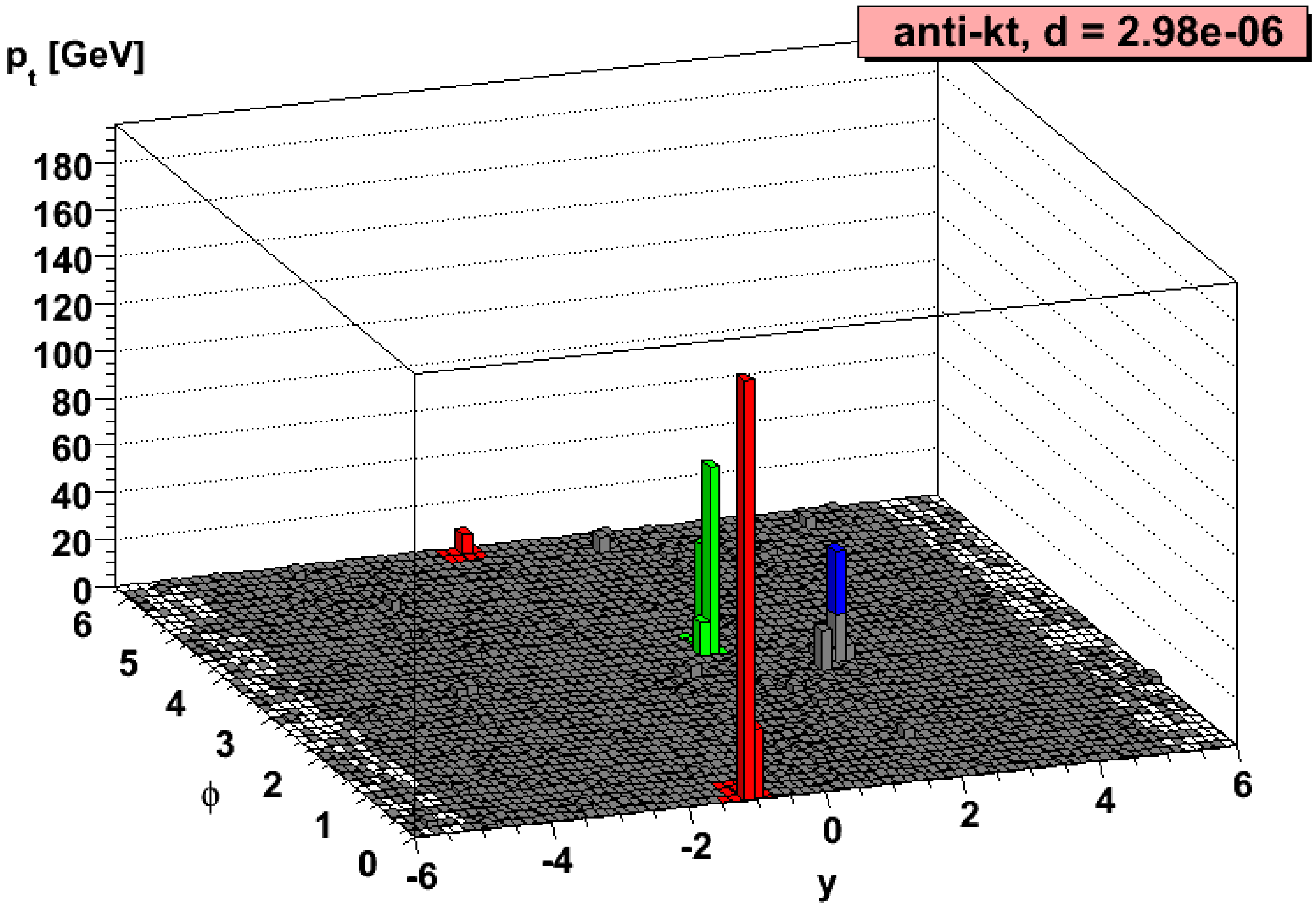

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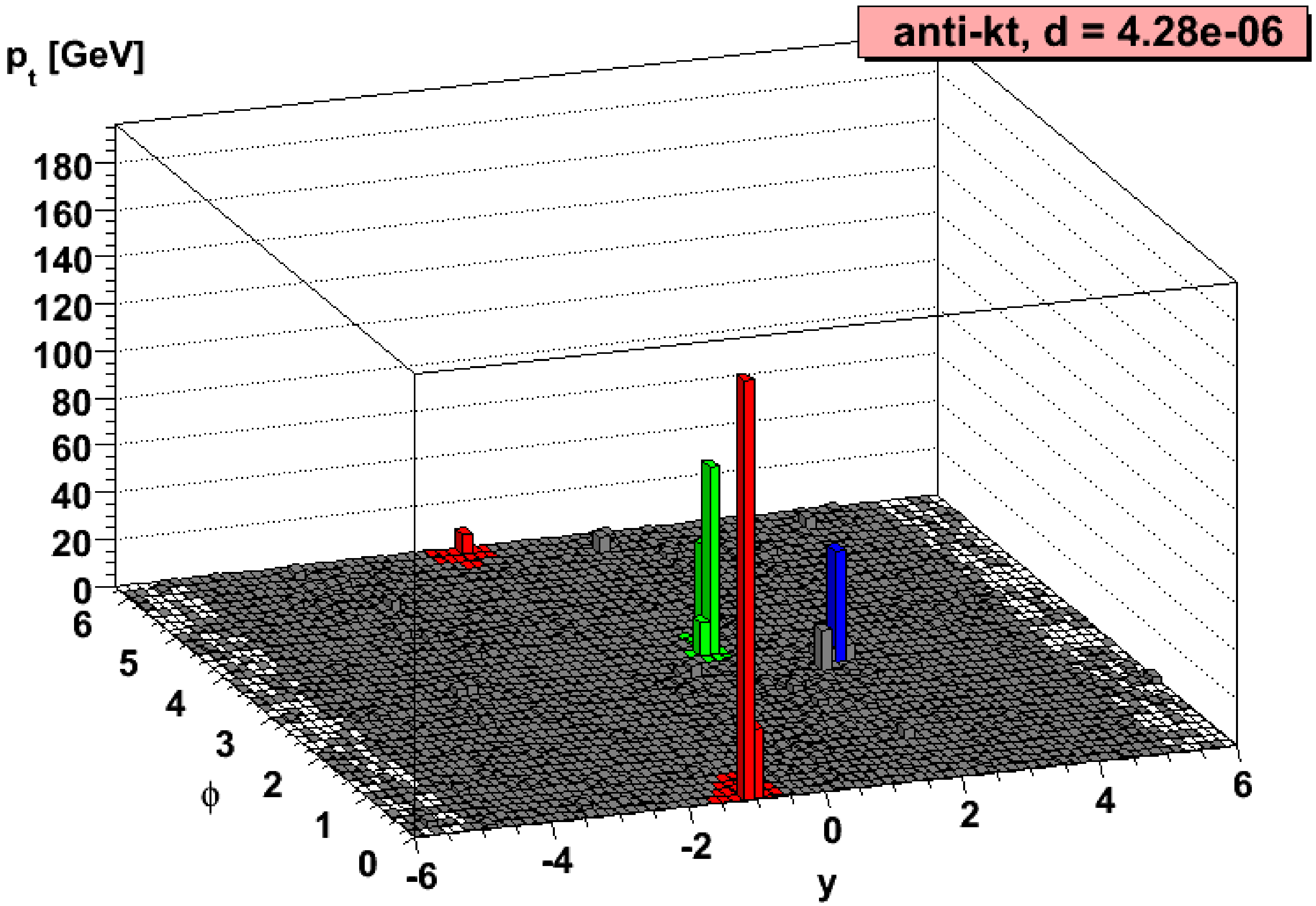

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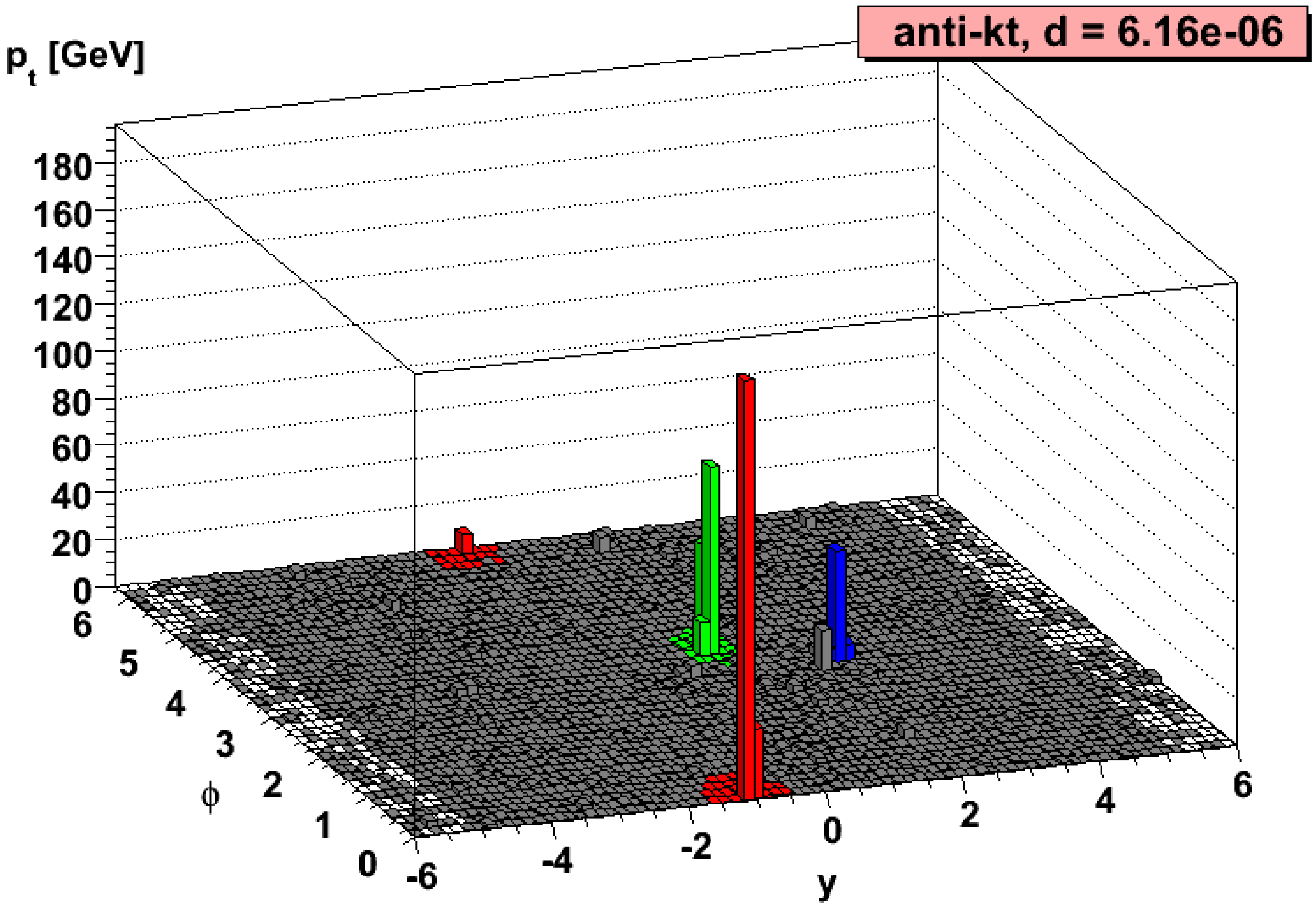

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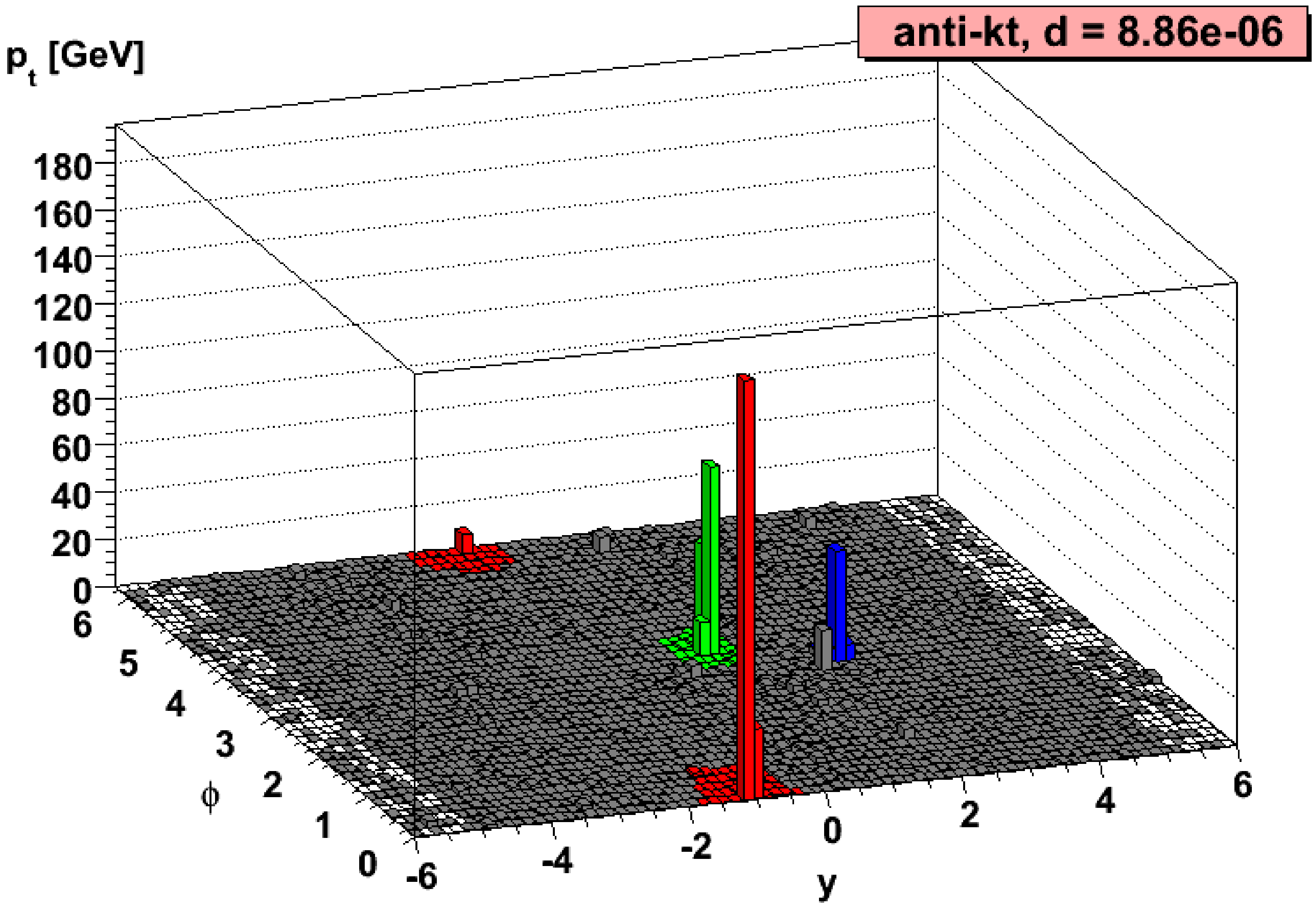

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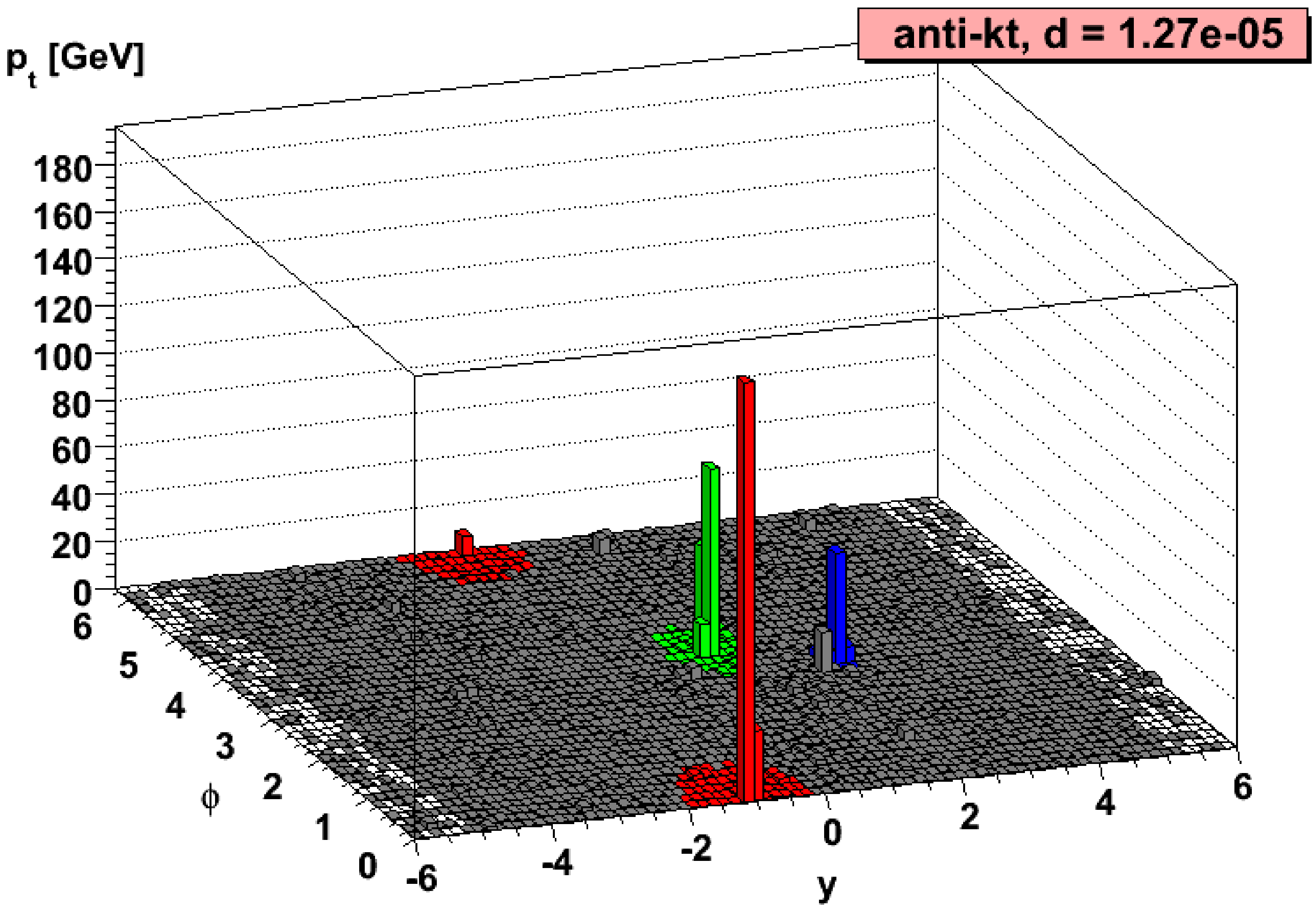

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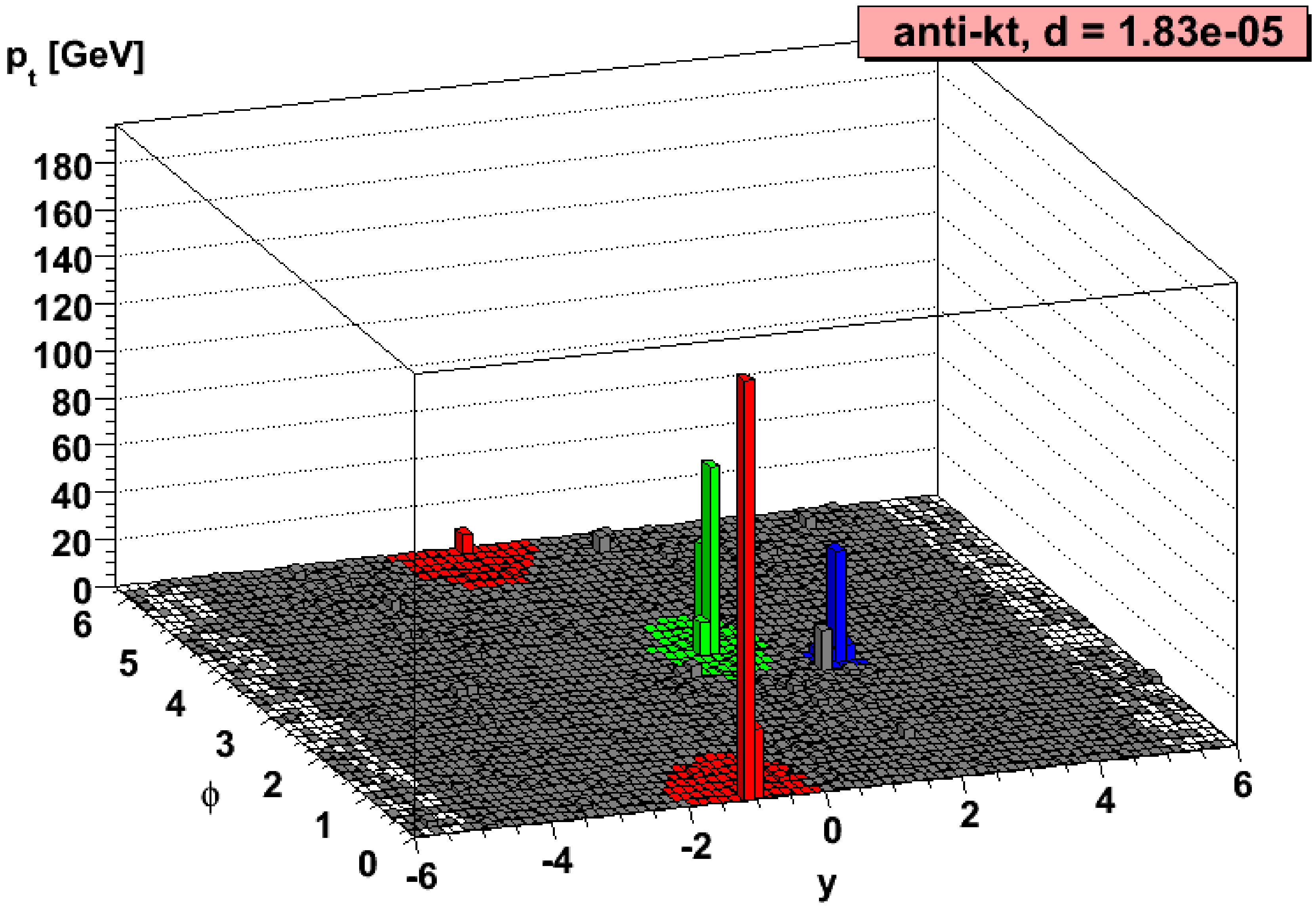

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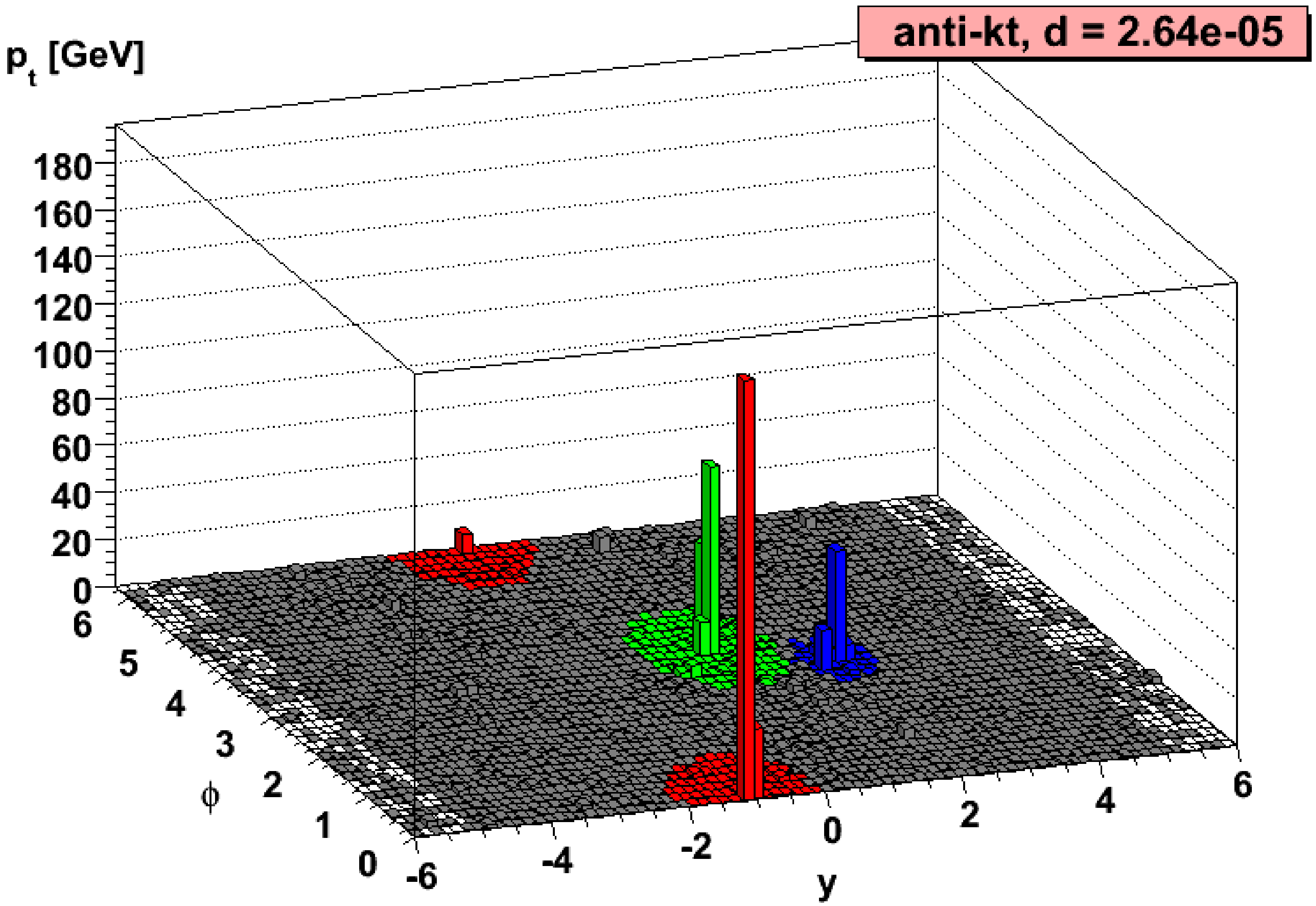

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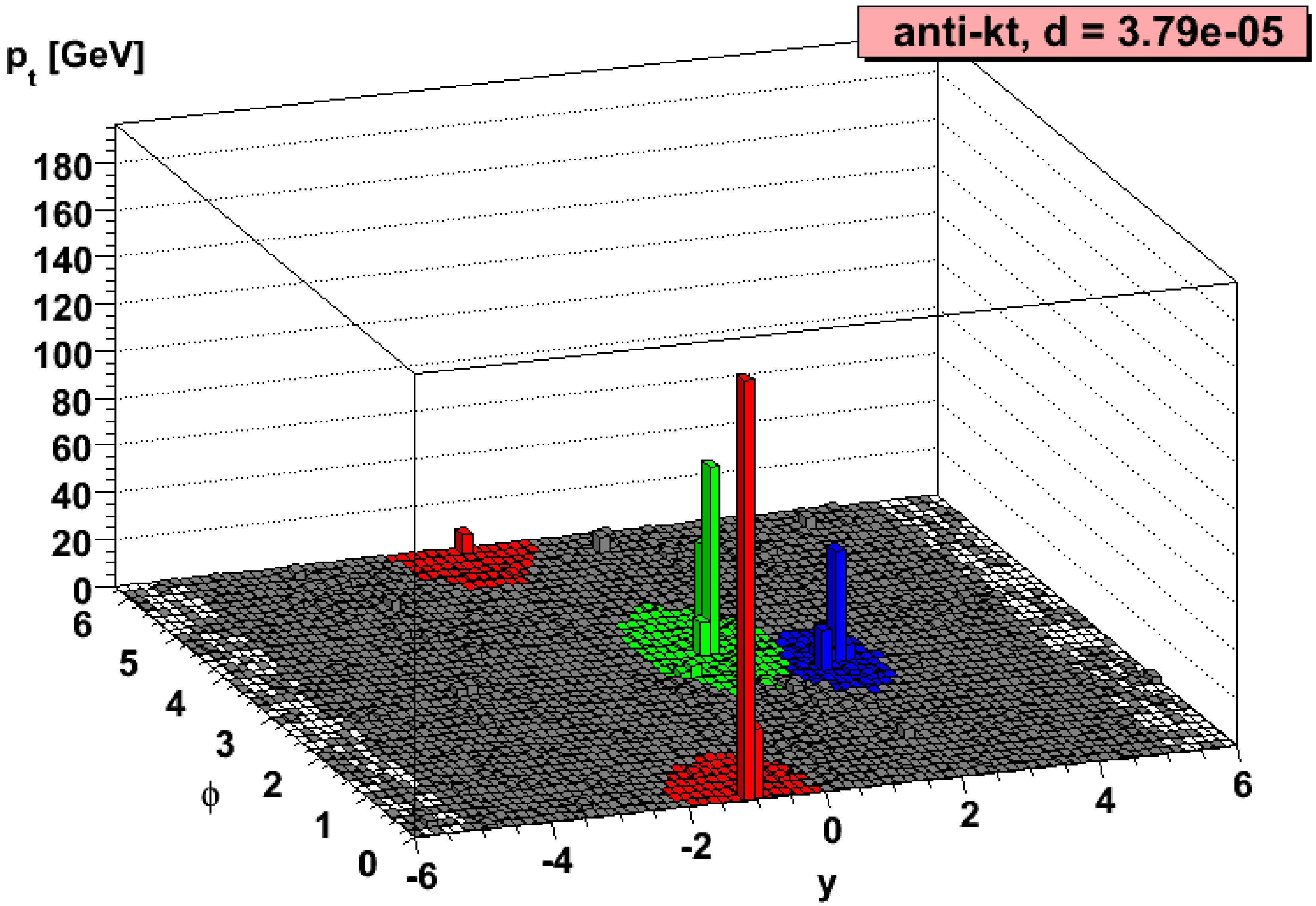

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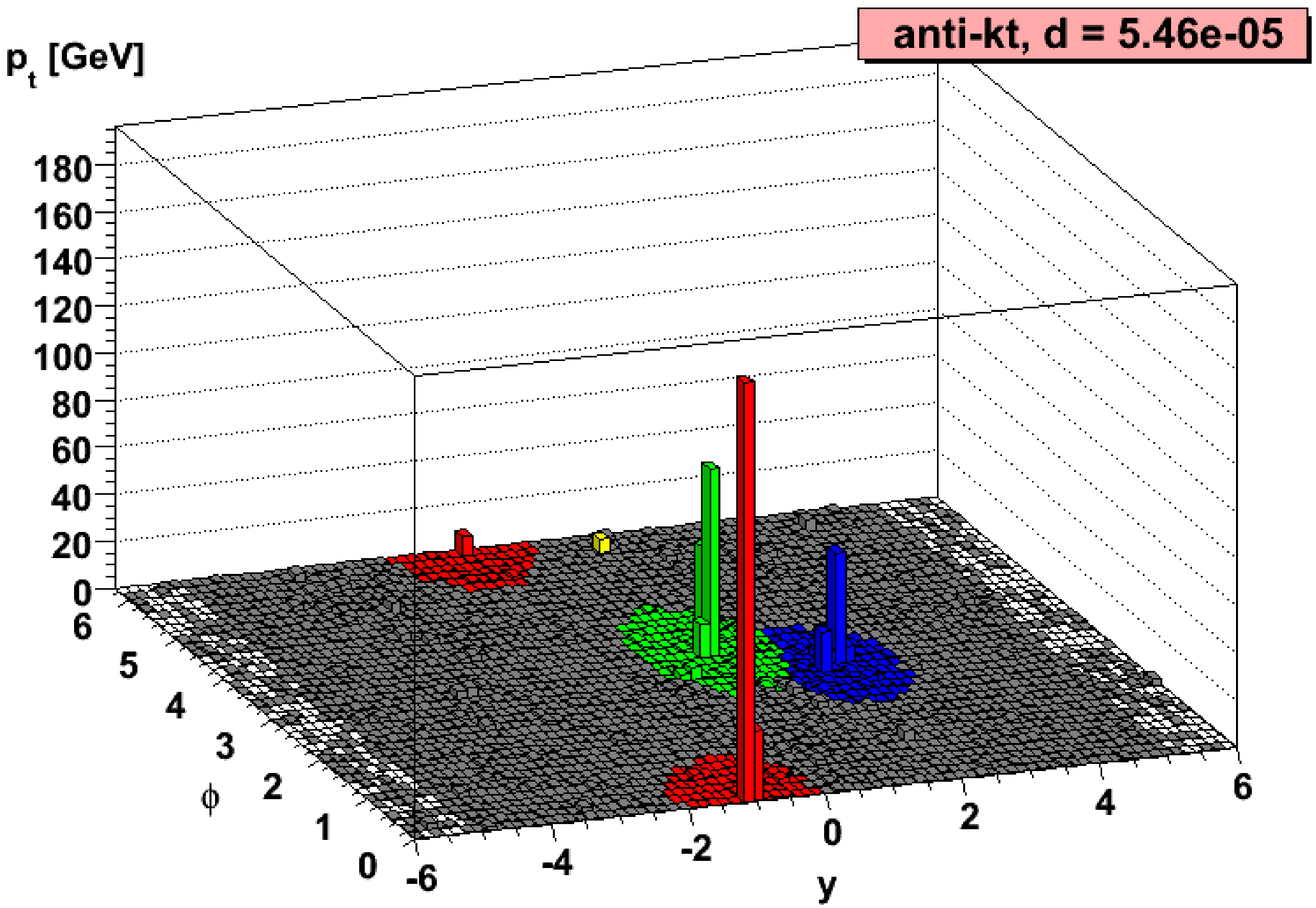

Anti-k$_t$ in action

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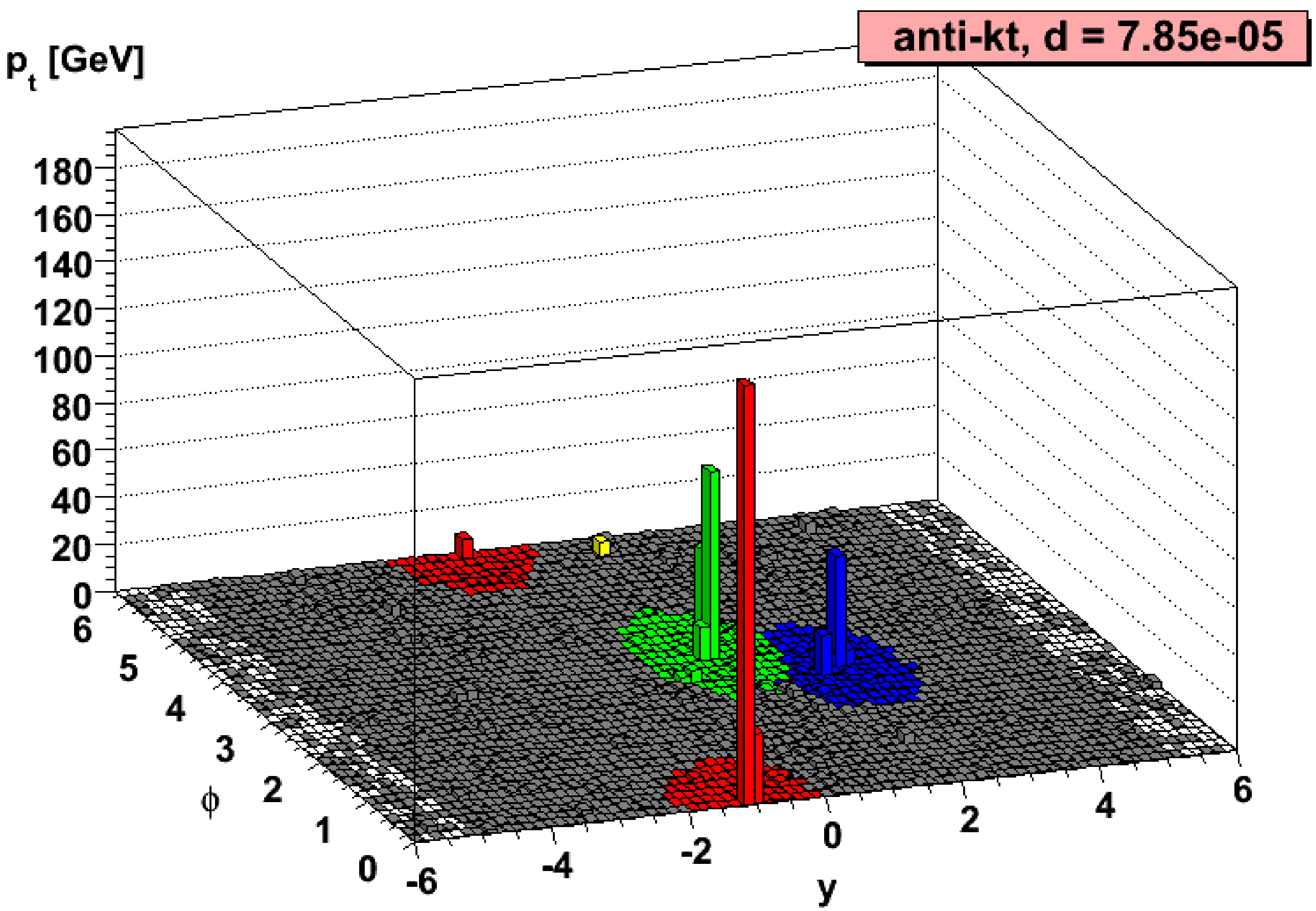

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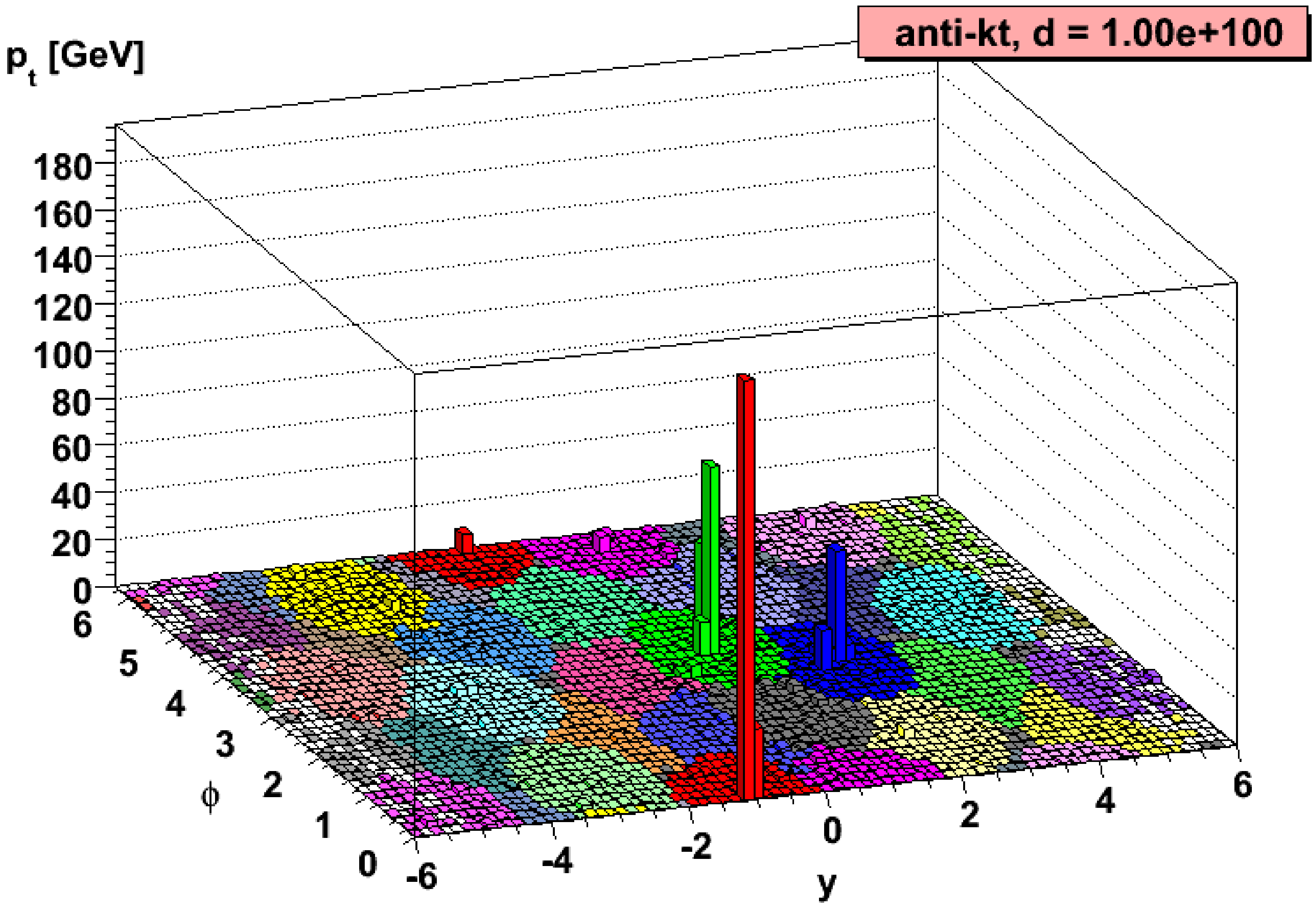
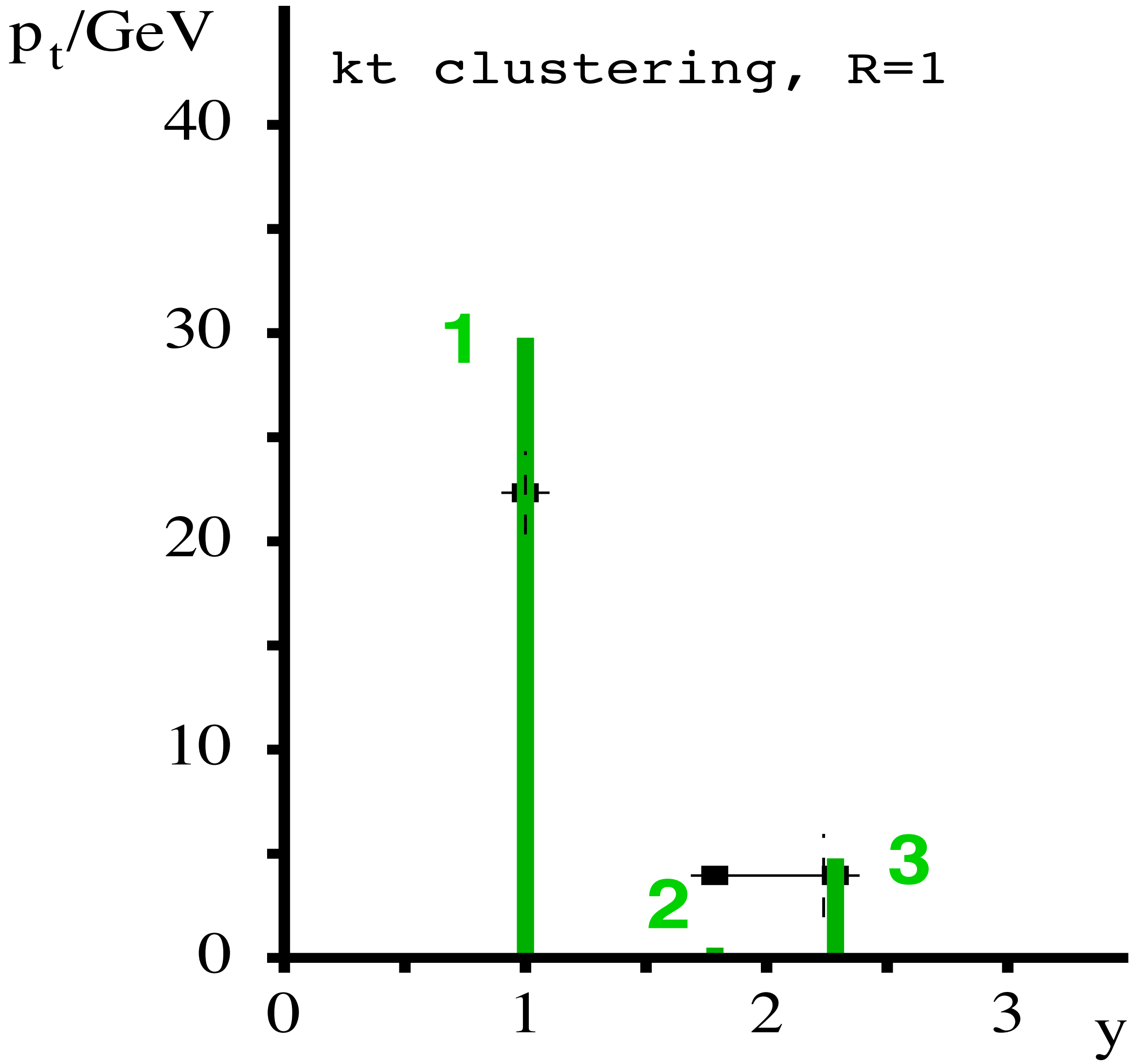
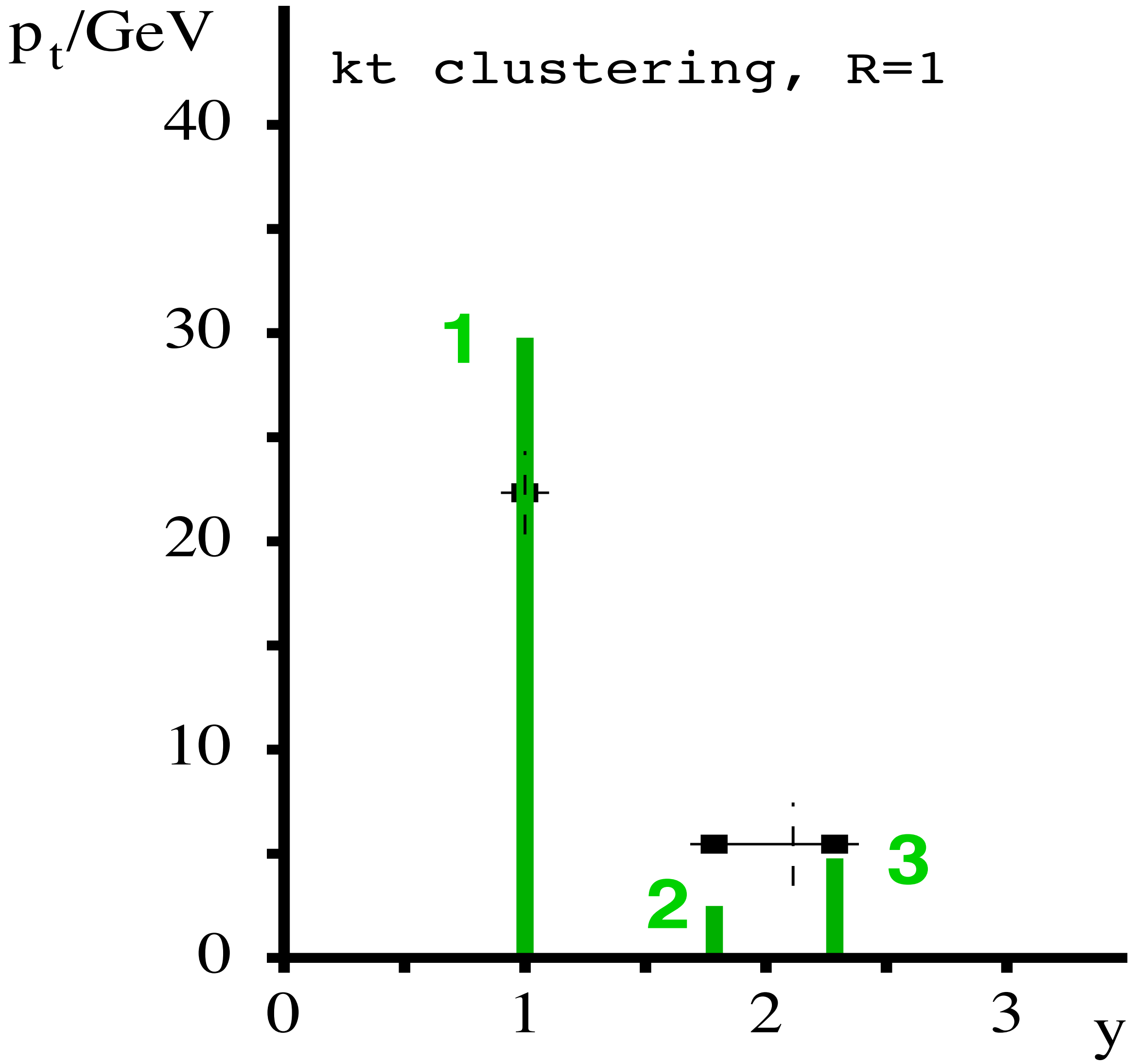


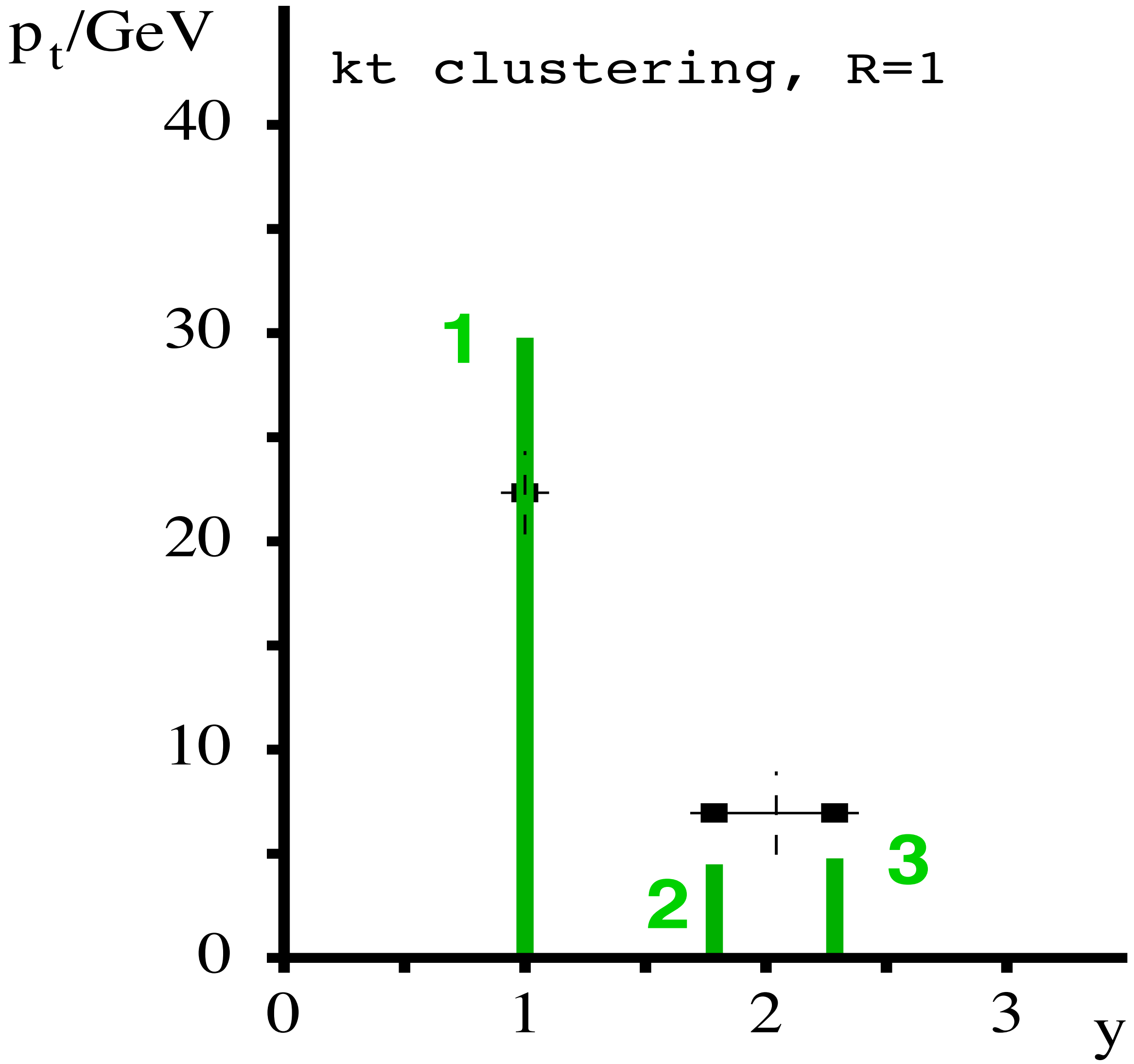
Anti-k_t gives circular jets ("cone-like") in a way that's infrared safe



p_t/GeV
kt clustering, R=1
40
30
20
10
0
1
2
3
0
1
2
3
y

Linearity: k_t v. anti-k_t

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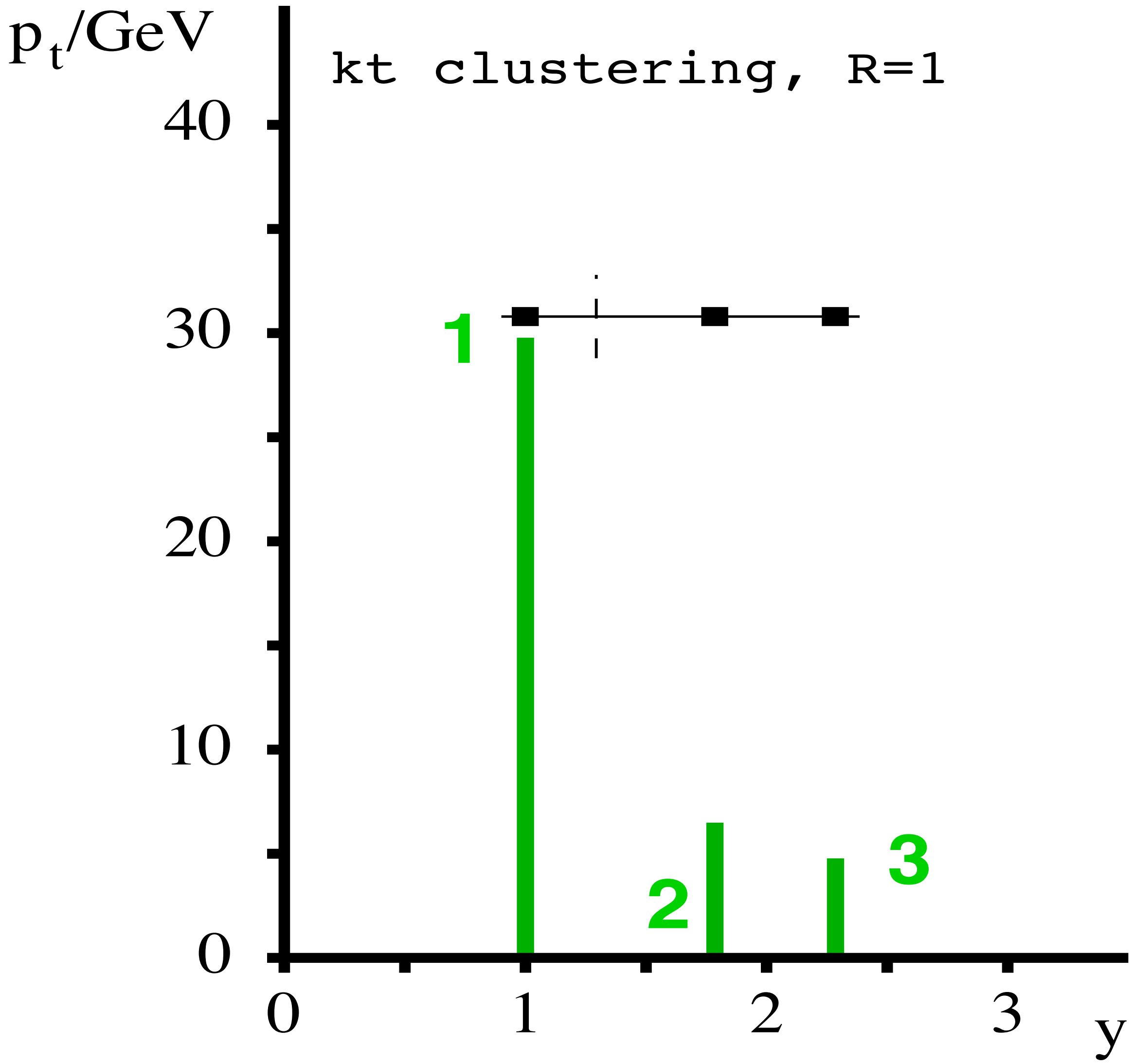

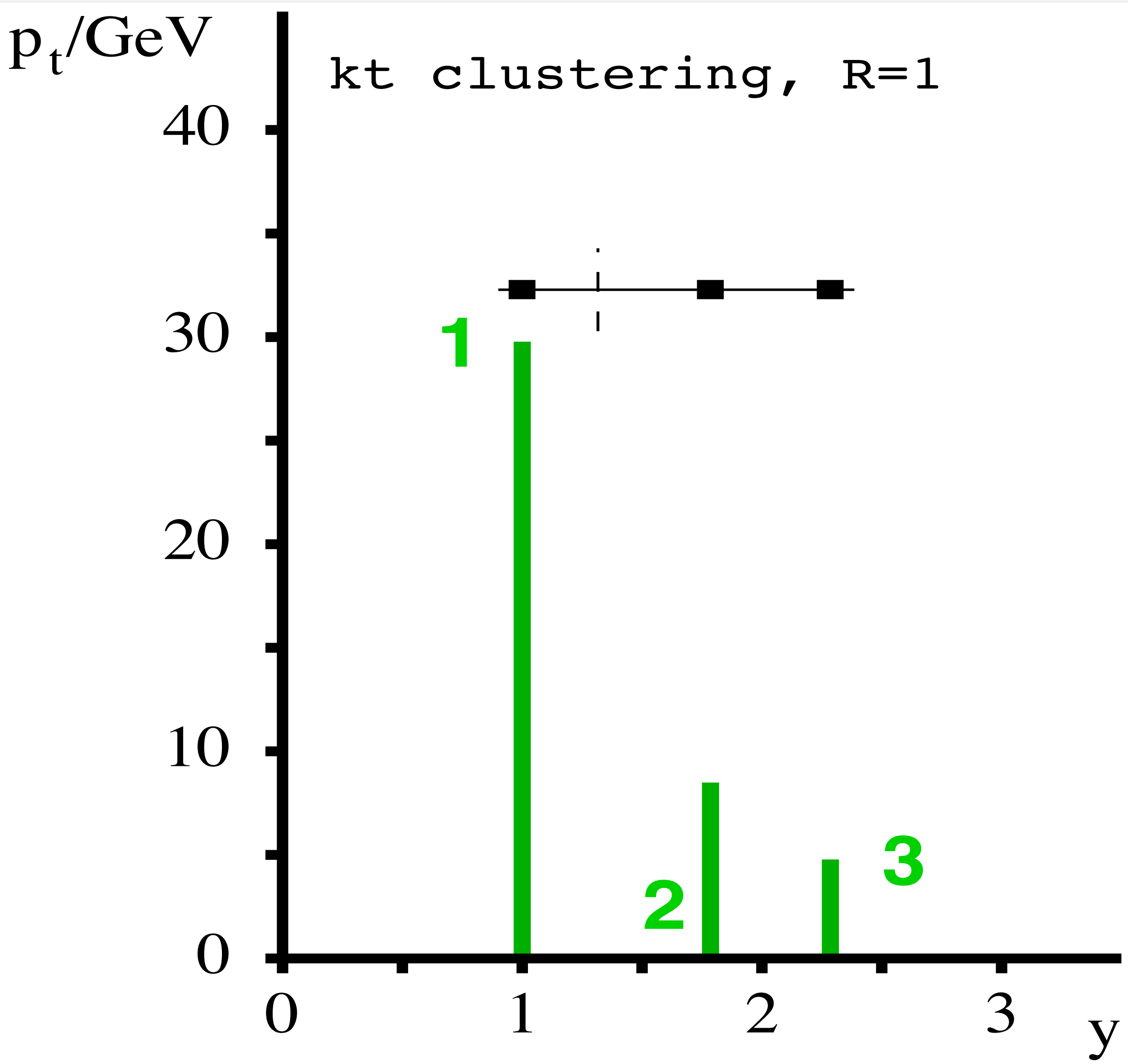
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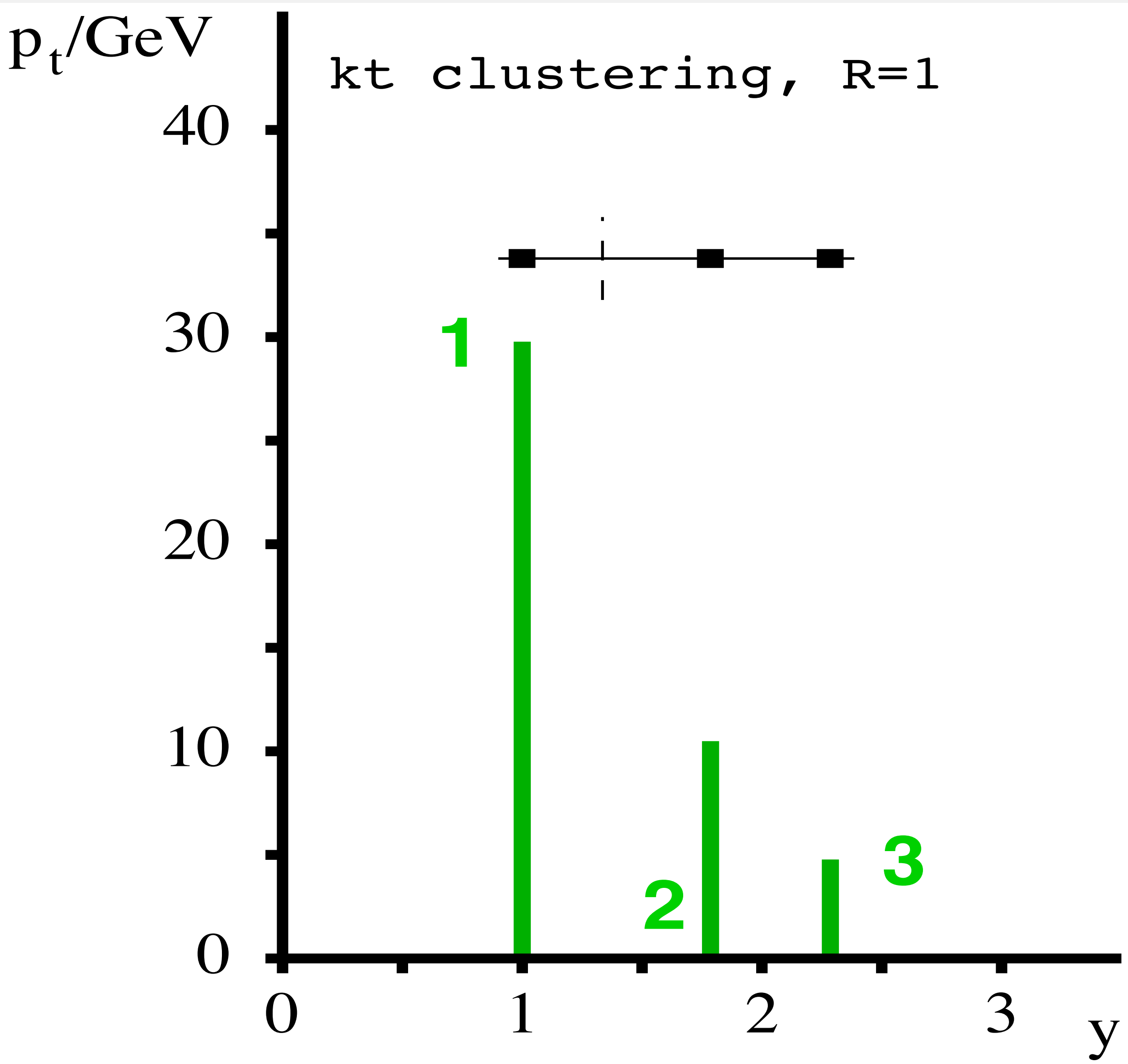

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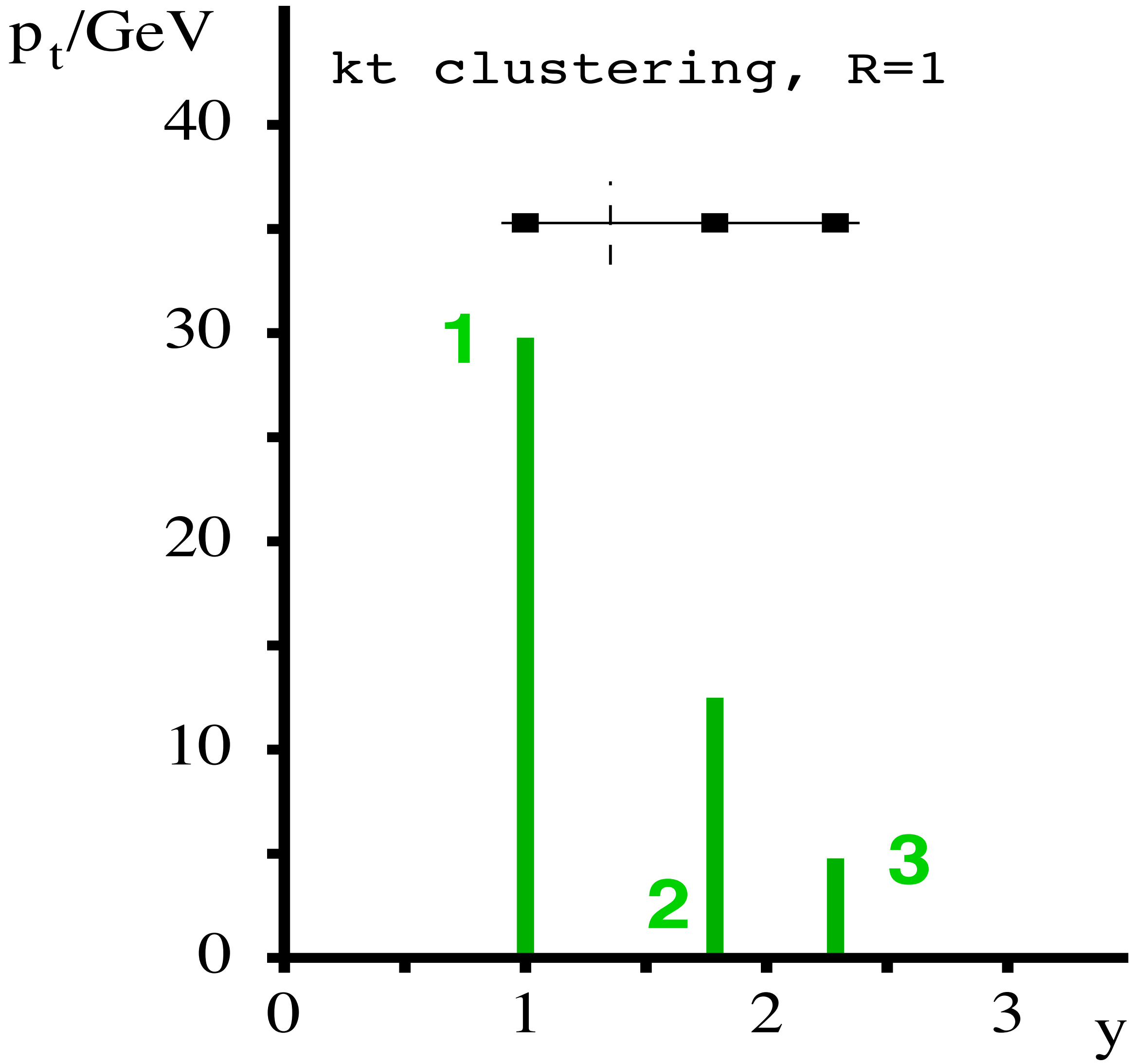
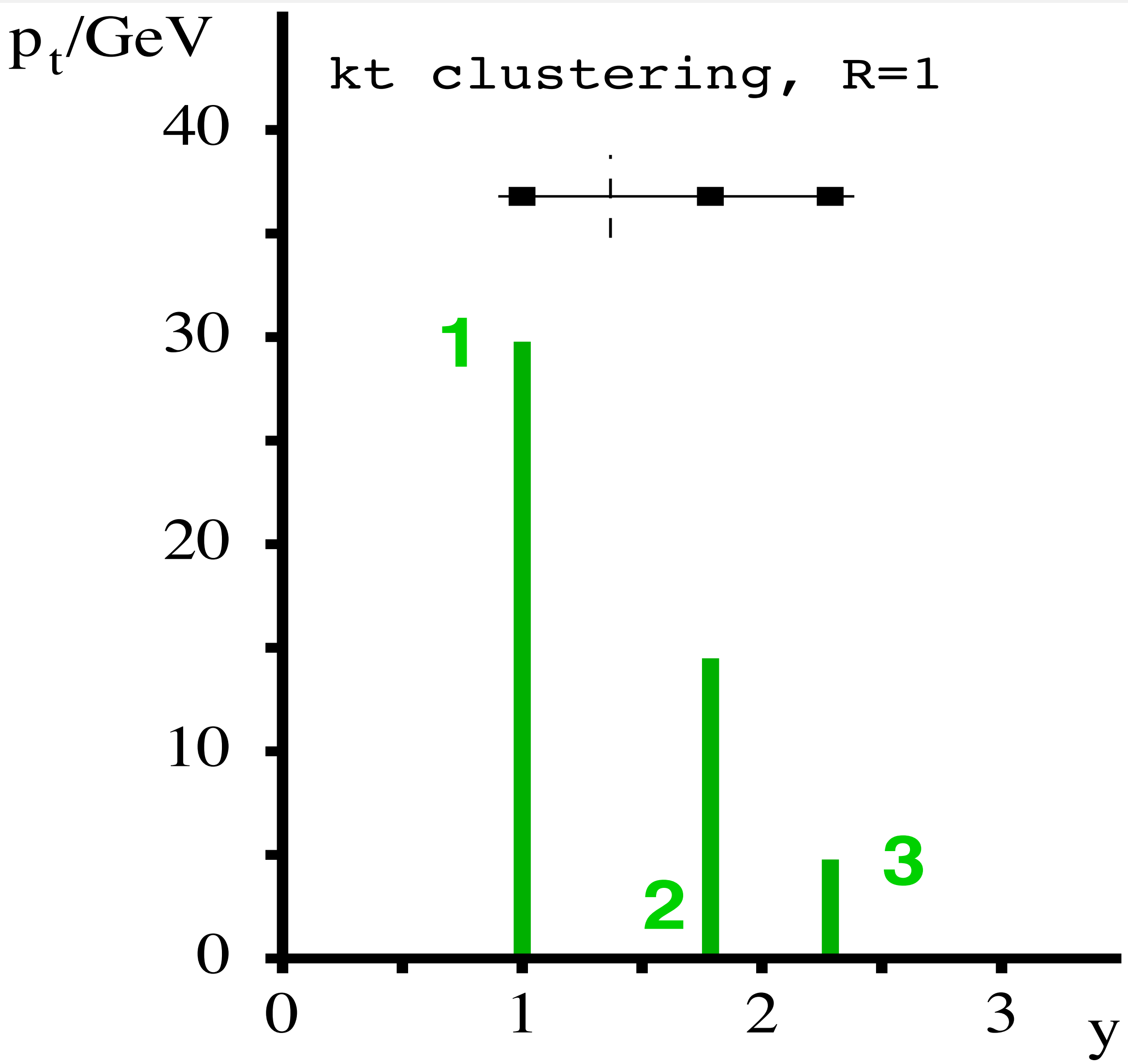
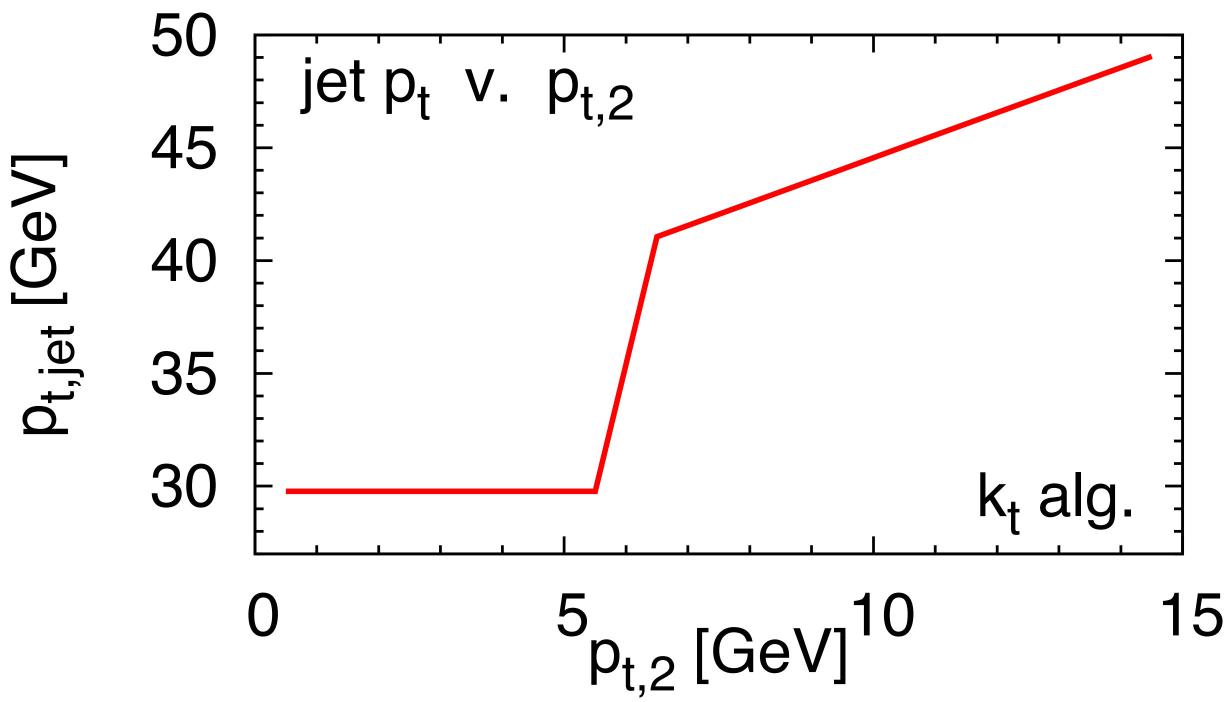
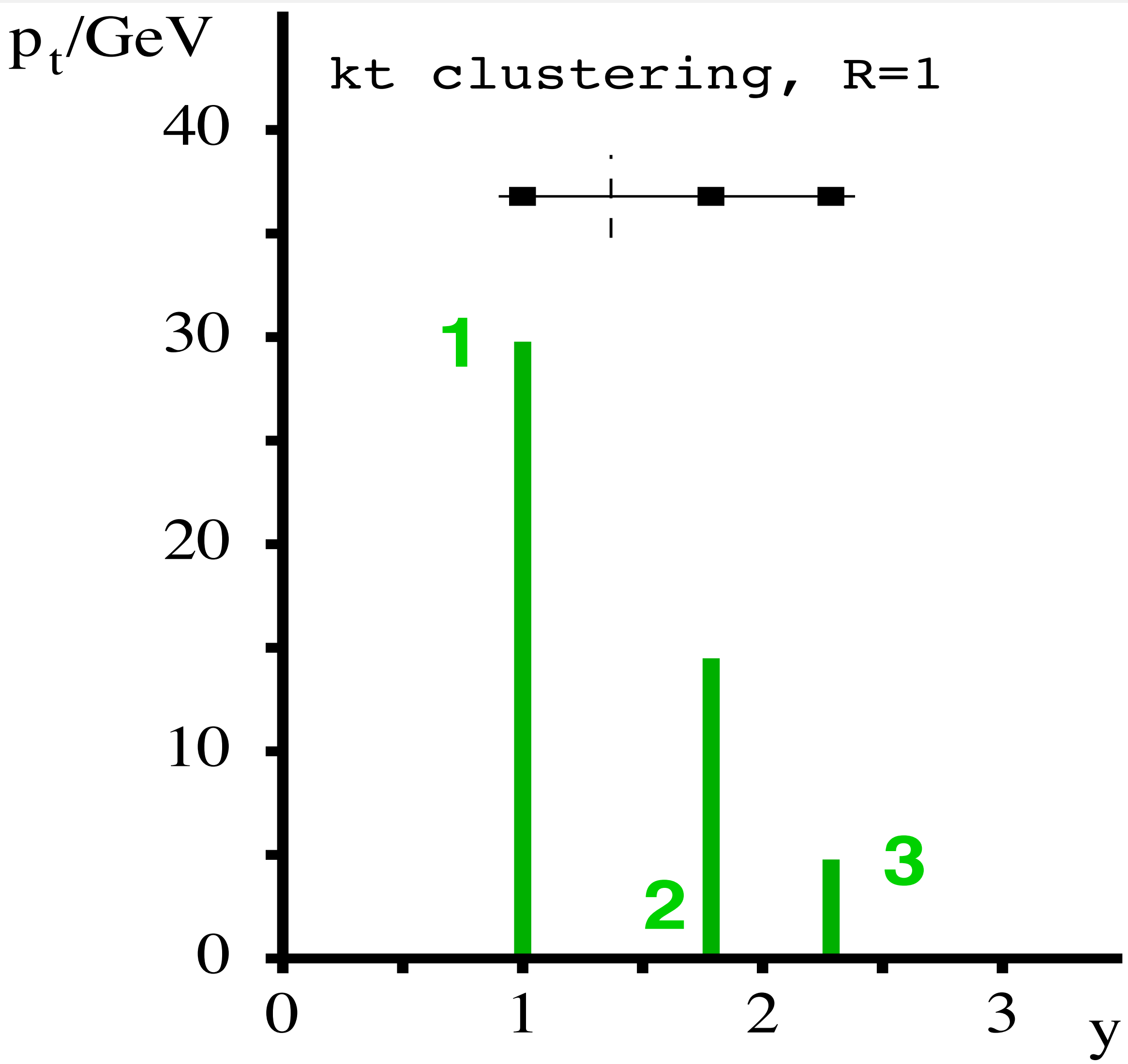


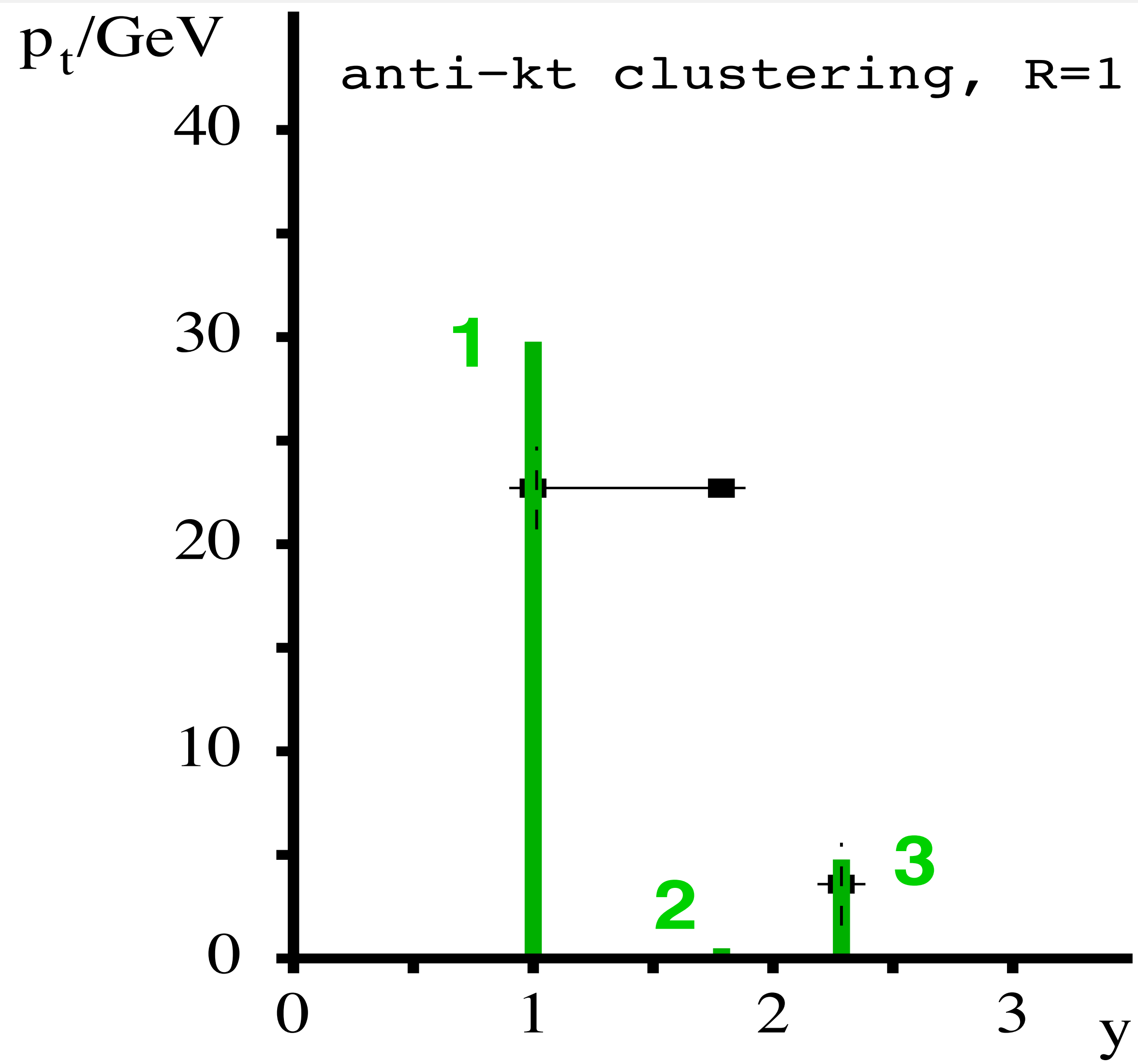
Linearity: k_t v. anti-k_t

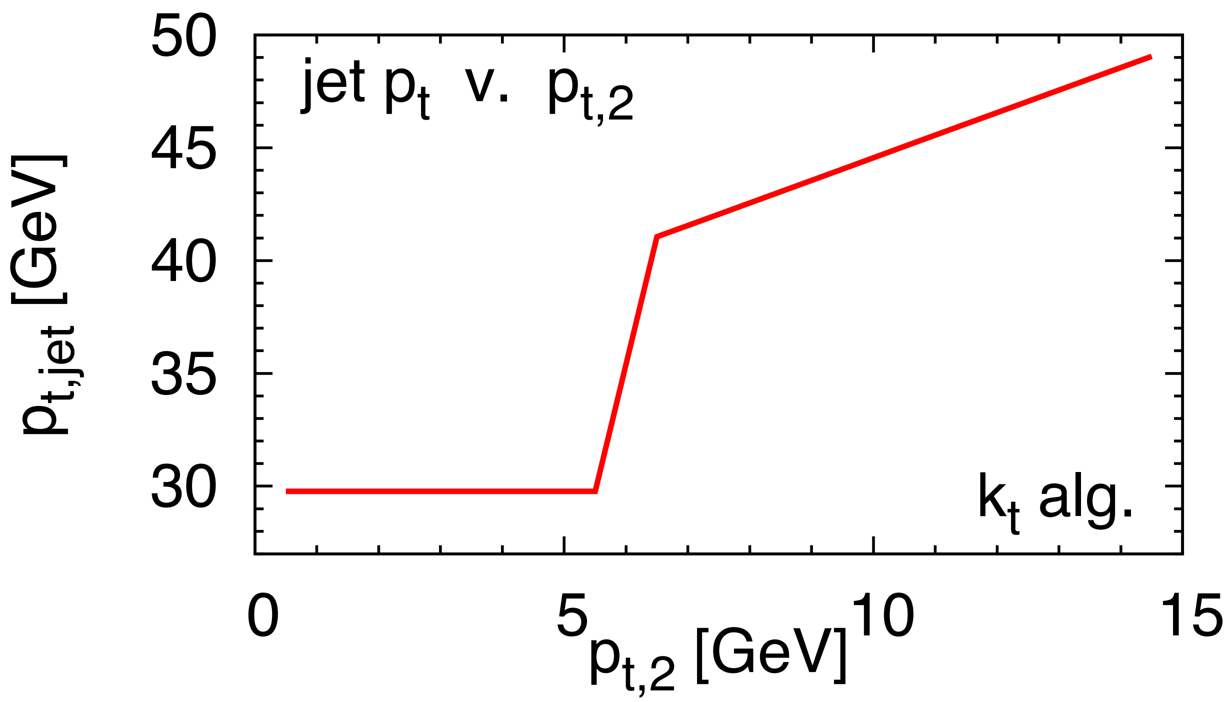


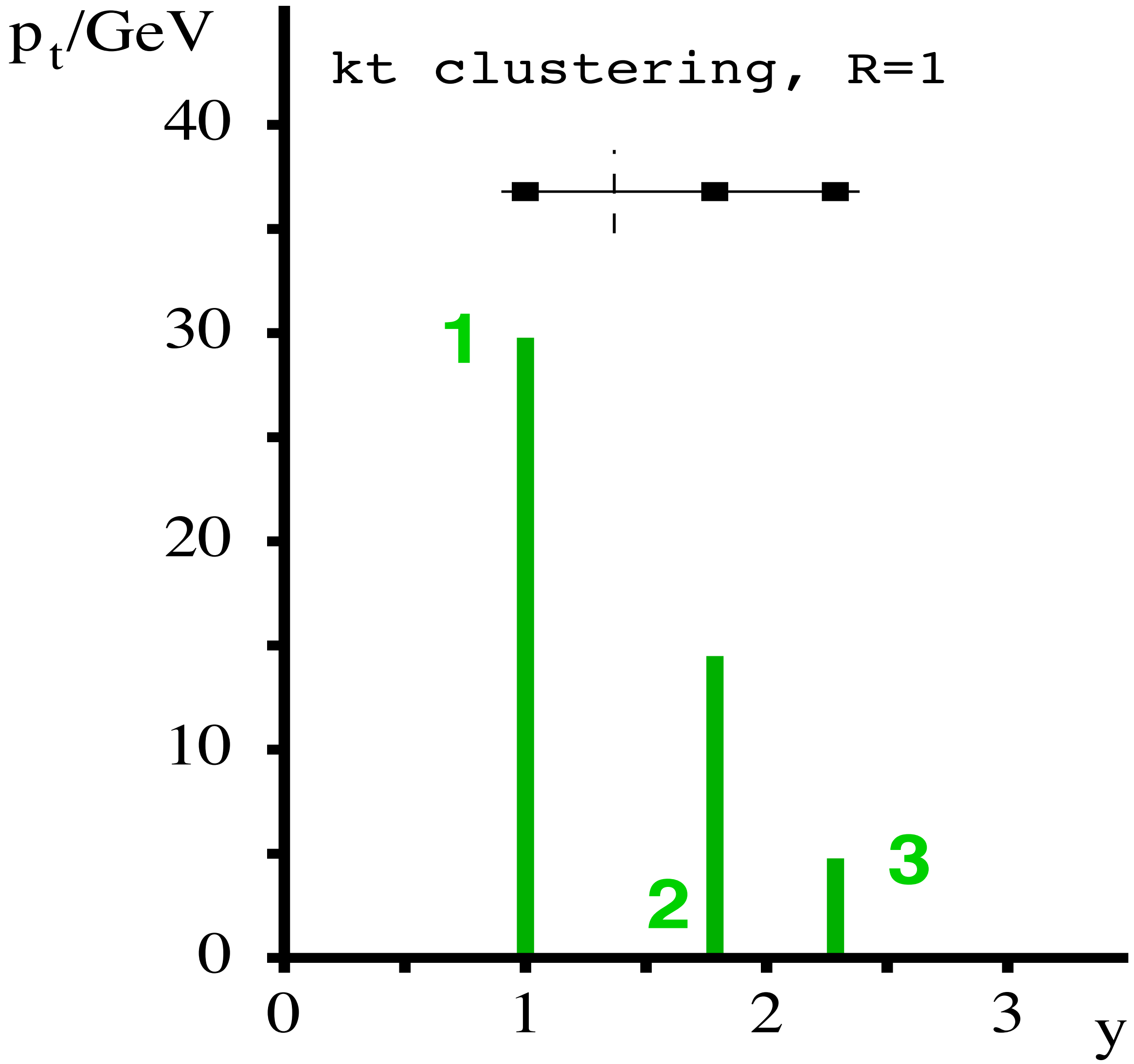


Linearity: k_t v. anti-k_t

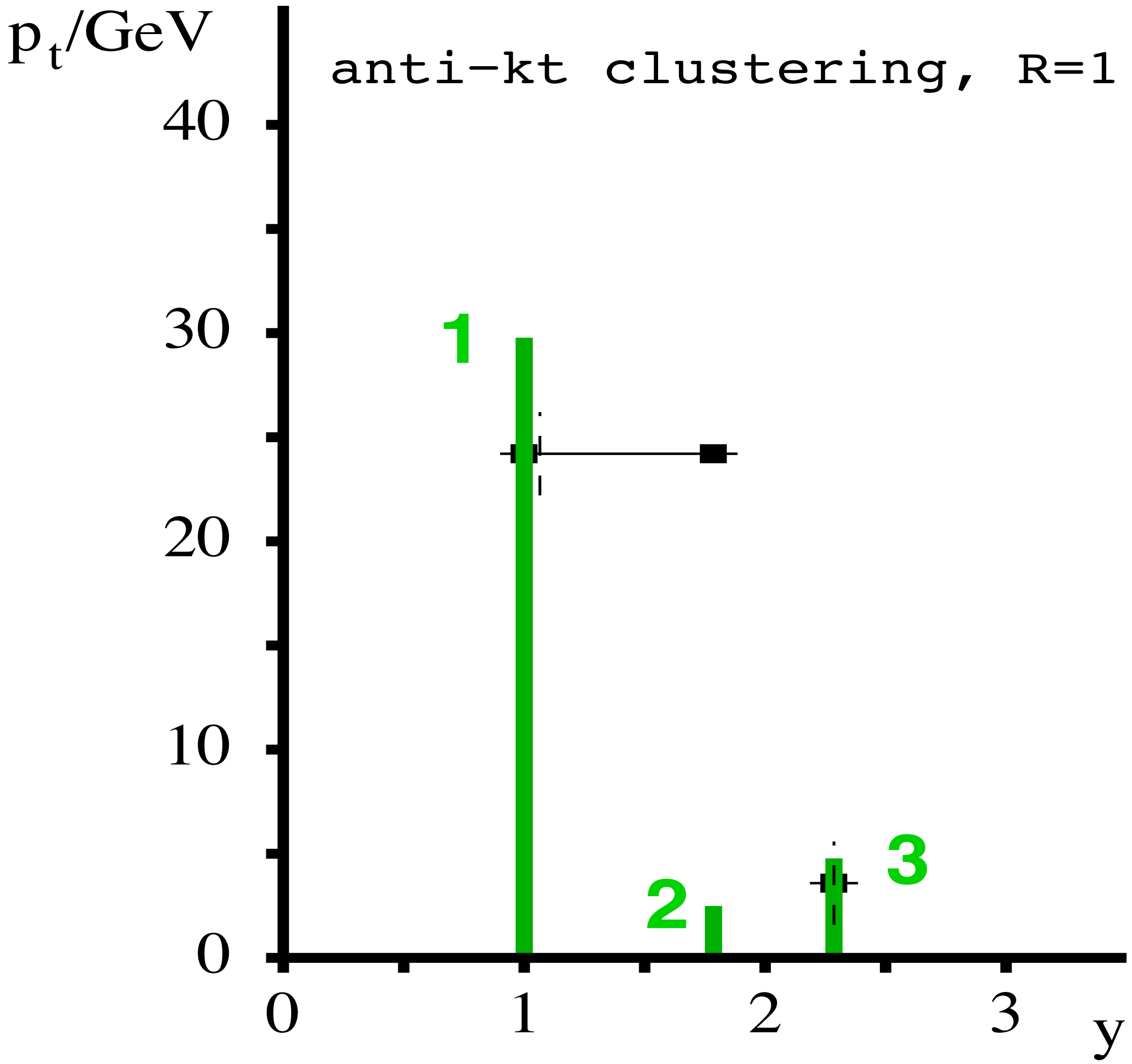




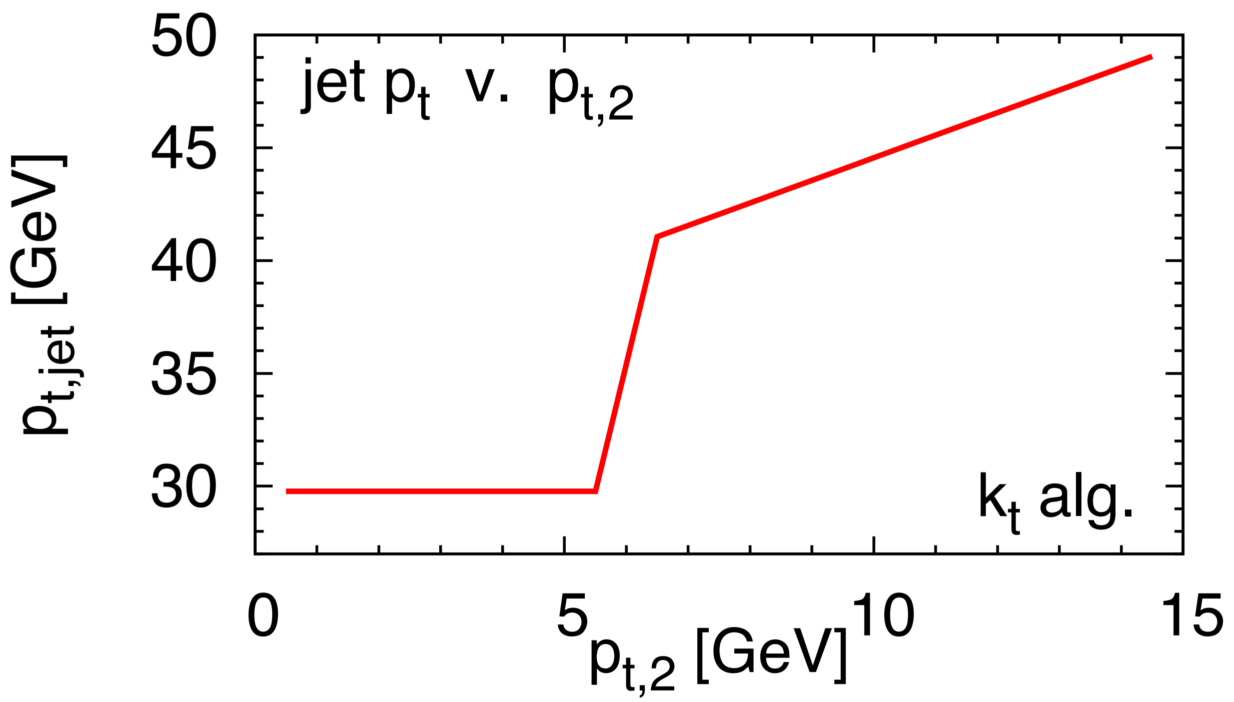




p_t/GeV
kt clustering, R=1
40
30
20
10
0
1
2
3
0
1
2
3
y

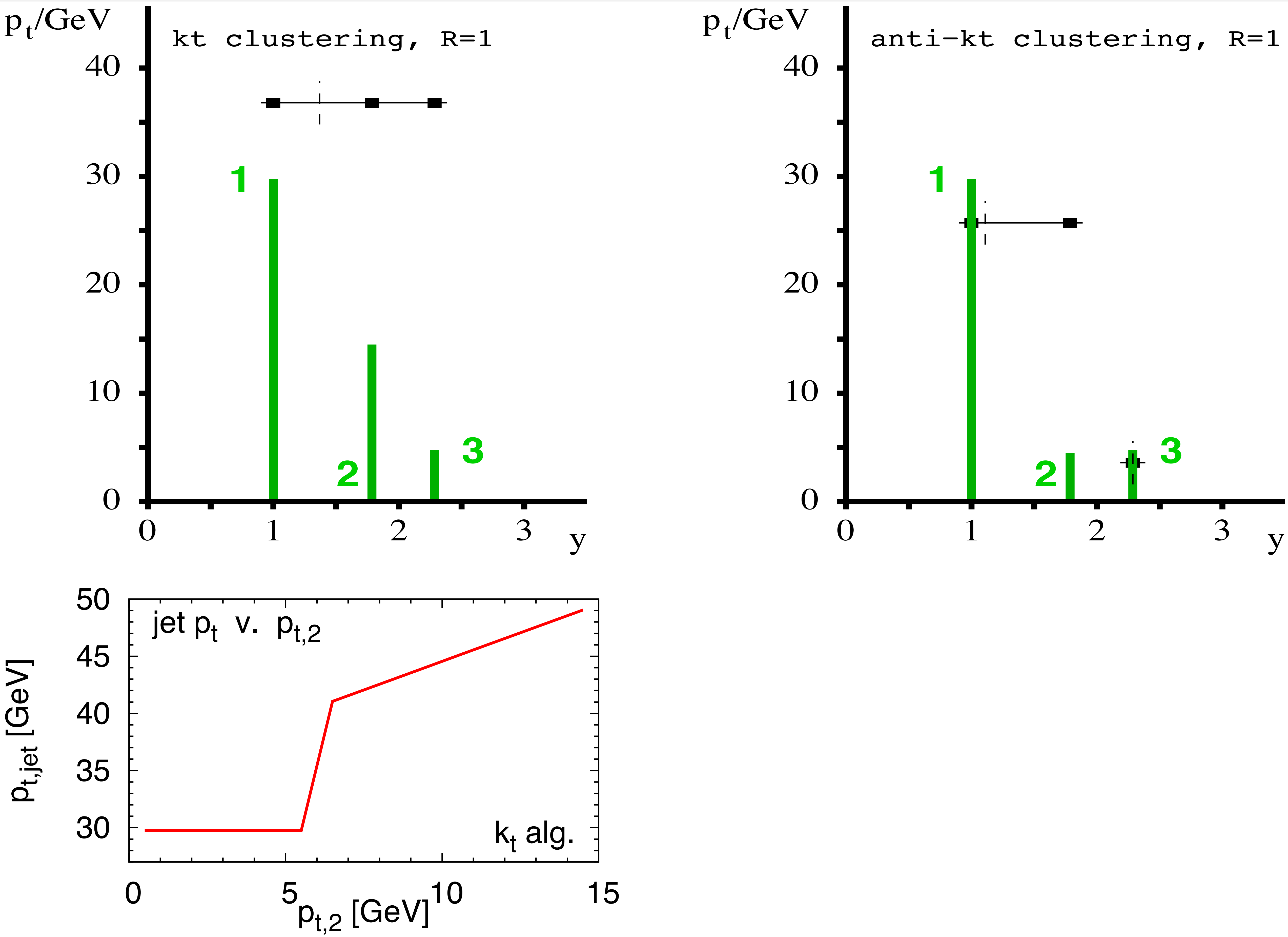


p_t/GeV
anti−kt clustering, R=1
40
30
20
10
0
1
2
3
0
1
2
3
y

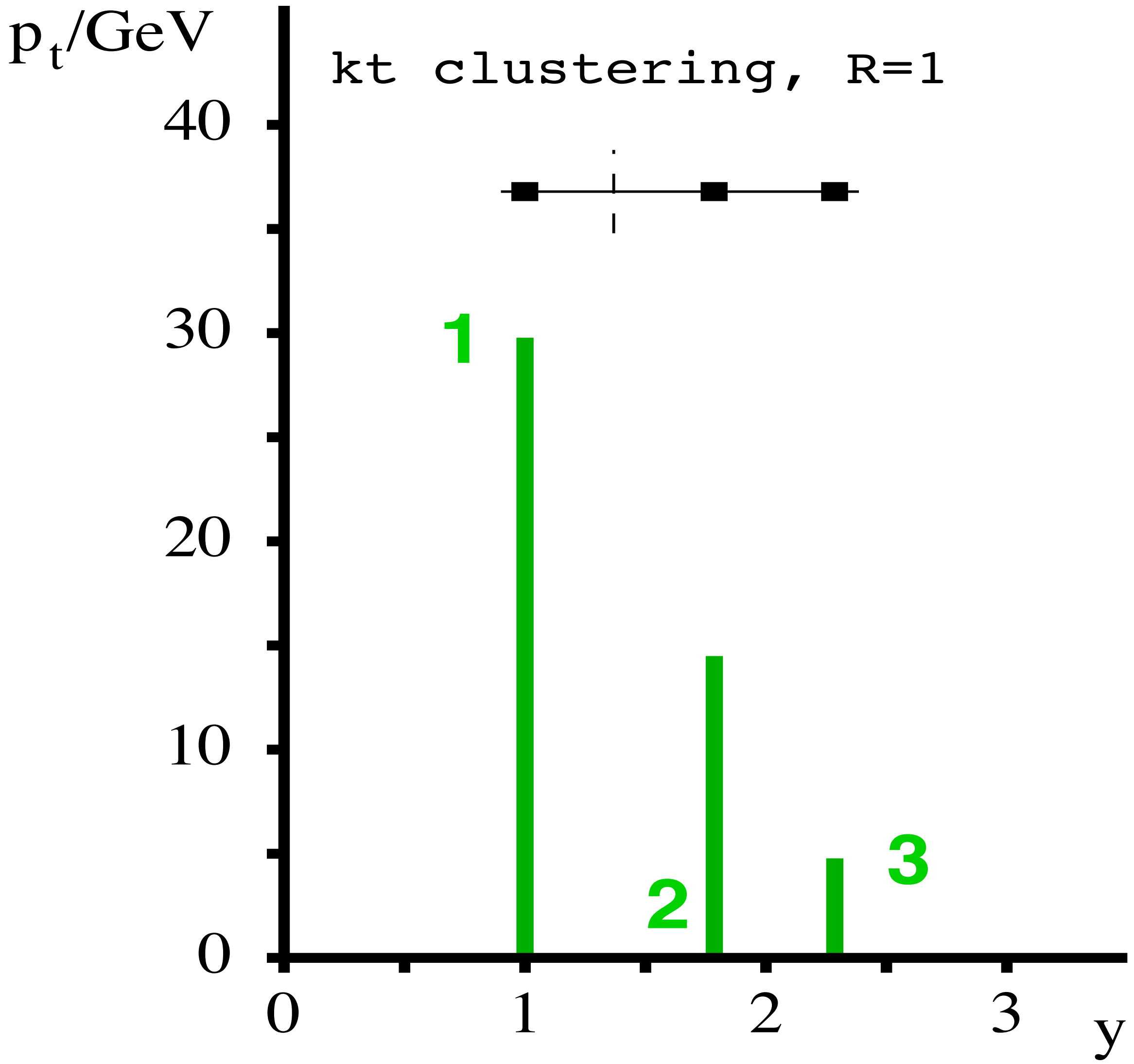


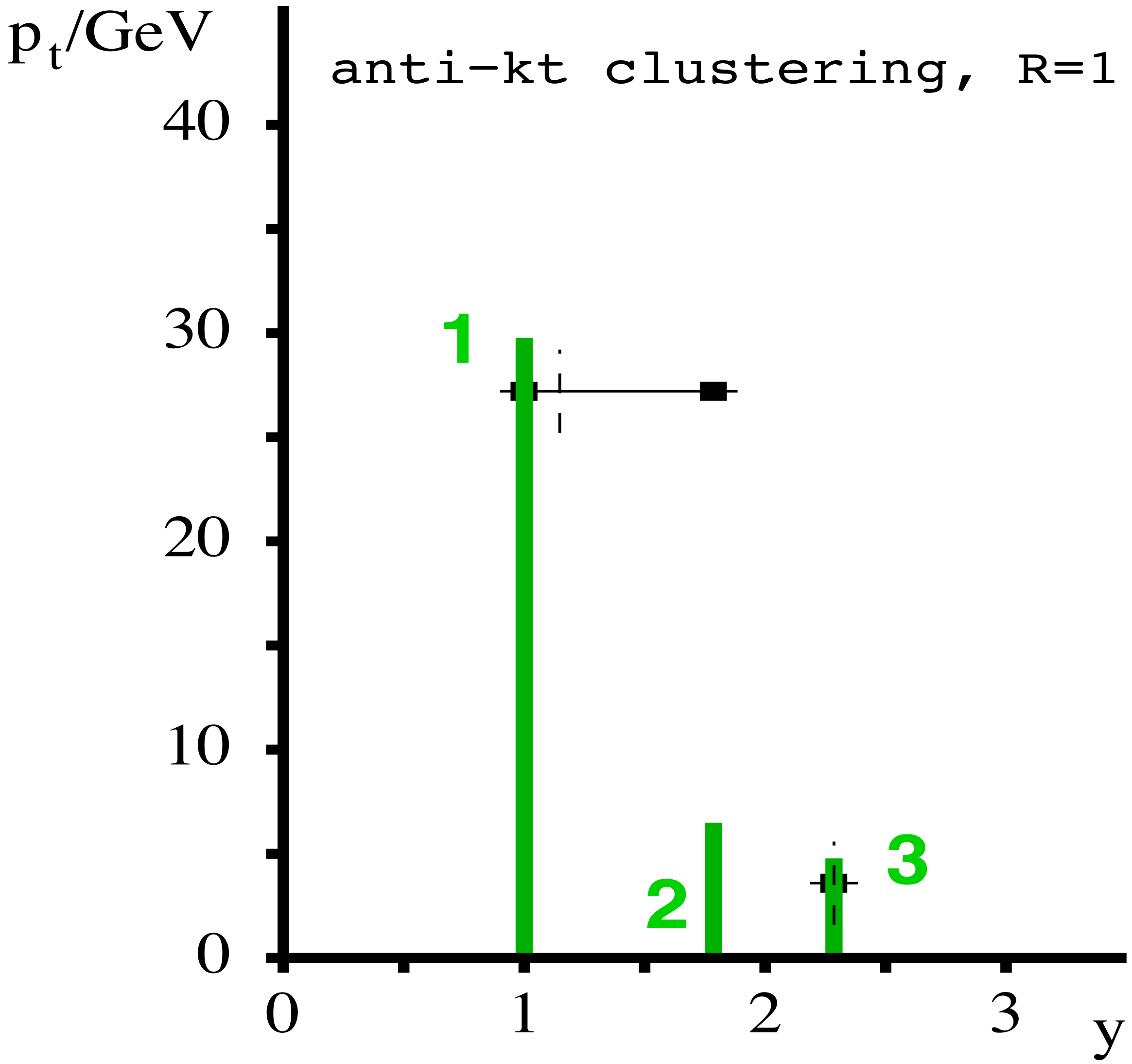
jet p_t v. p_t,2
k_t alg.
p_t,jet [GeV]
50
45
40
35
30
0
5
10
15
p_t,2 [GeV]

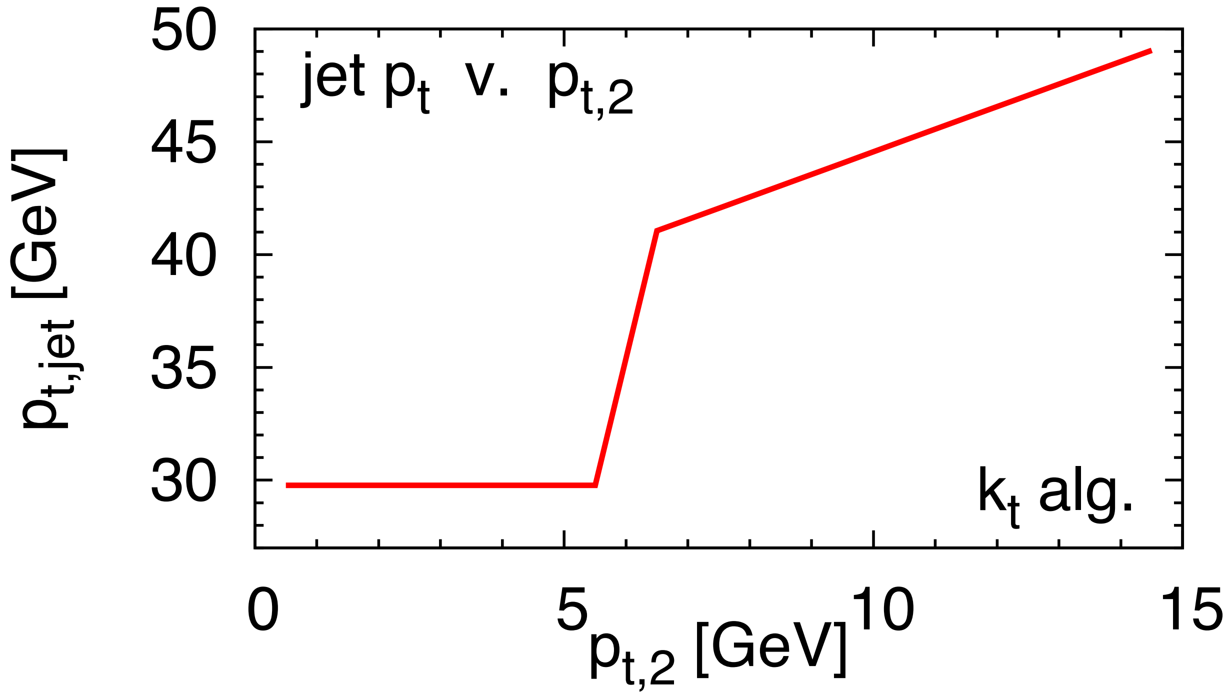
Linearity: k_t v. anti-k_t



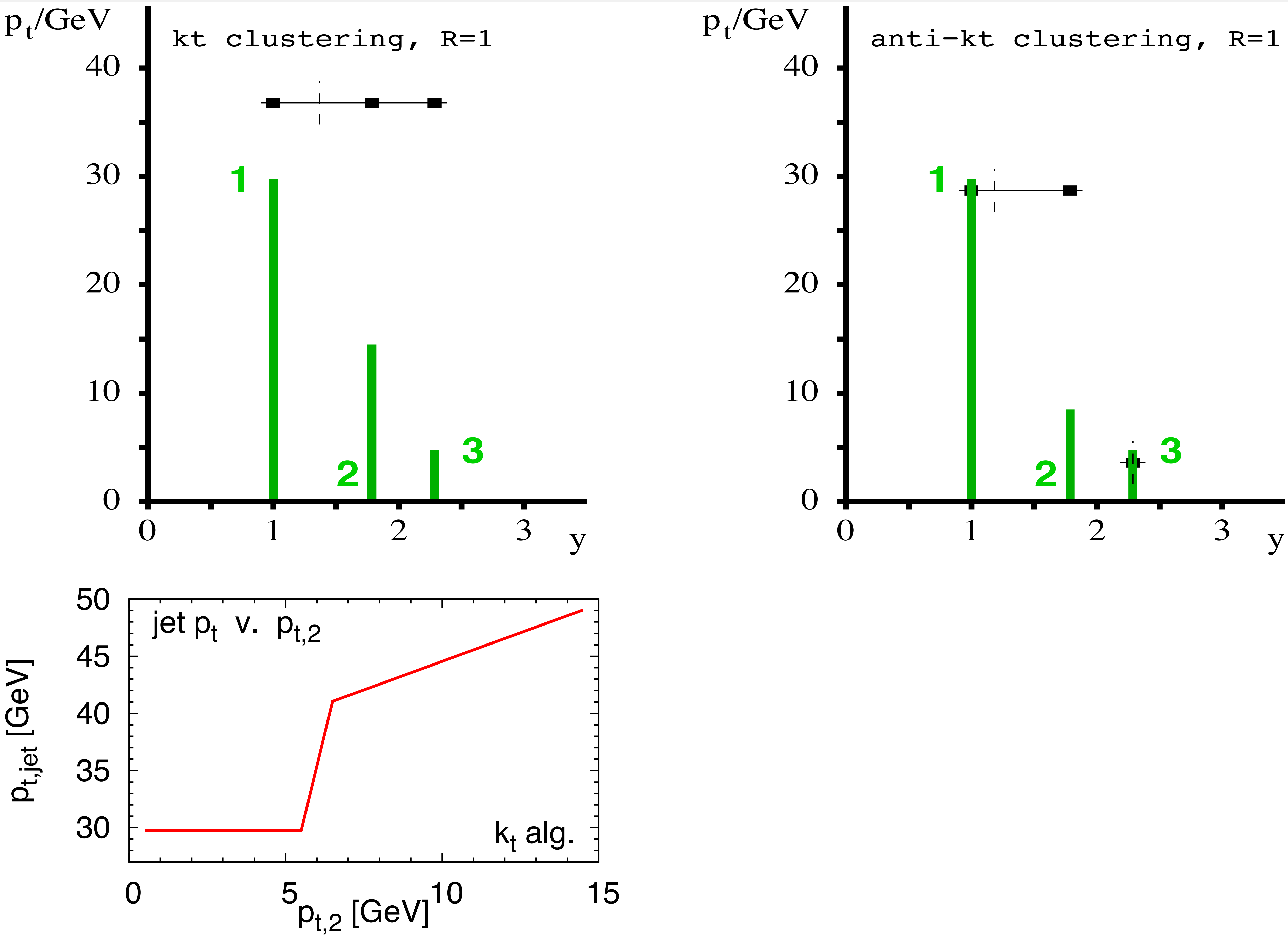
Linearity: k_t v. anti-k_t



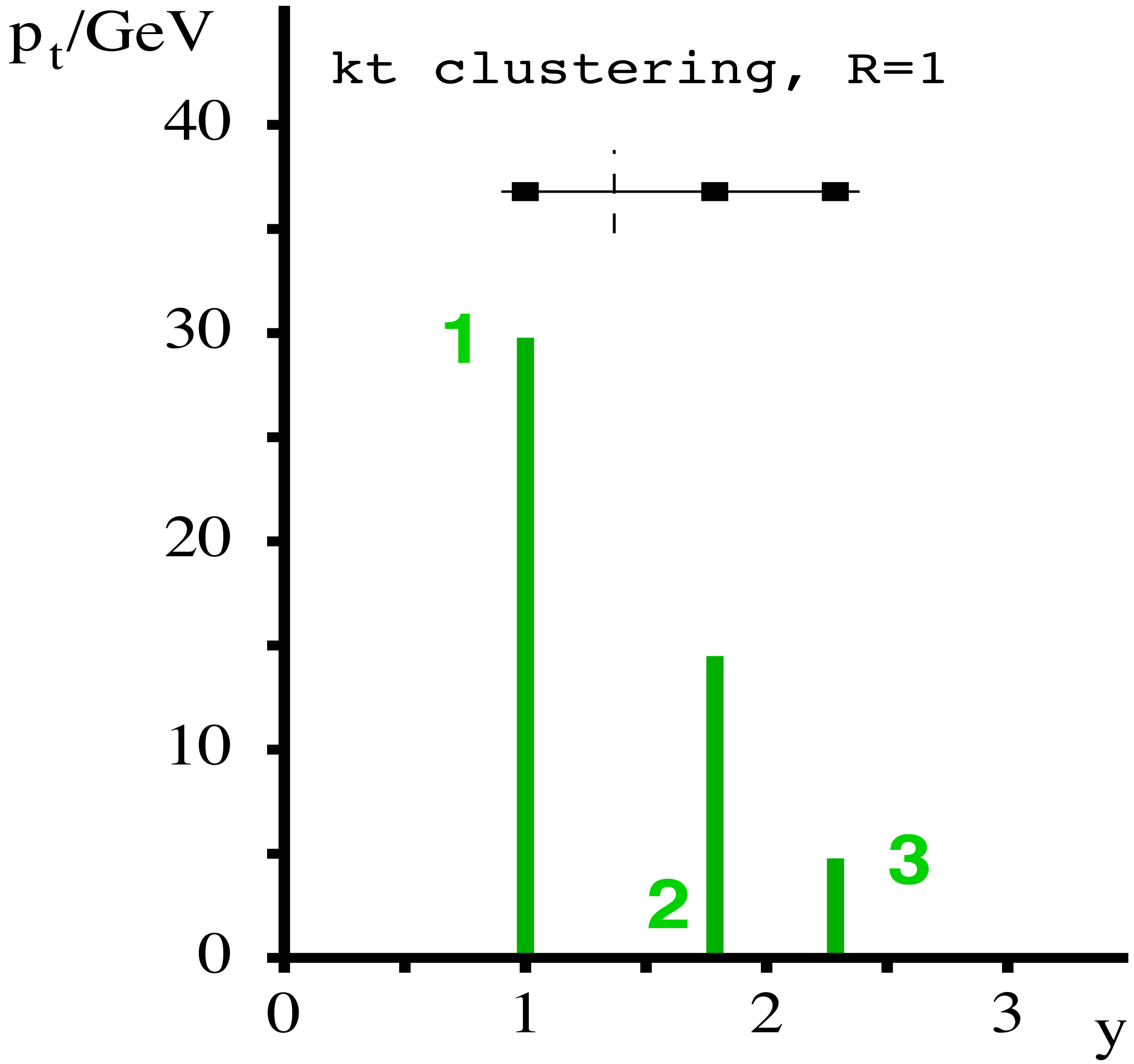


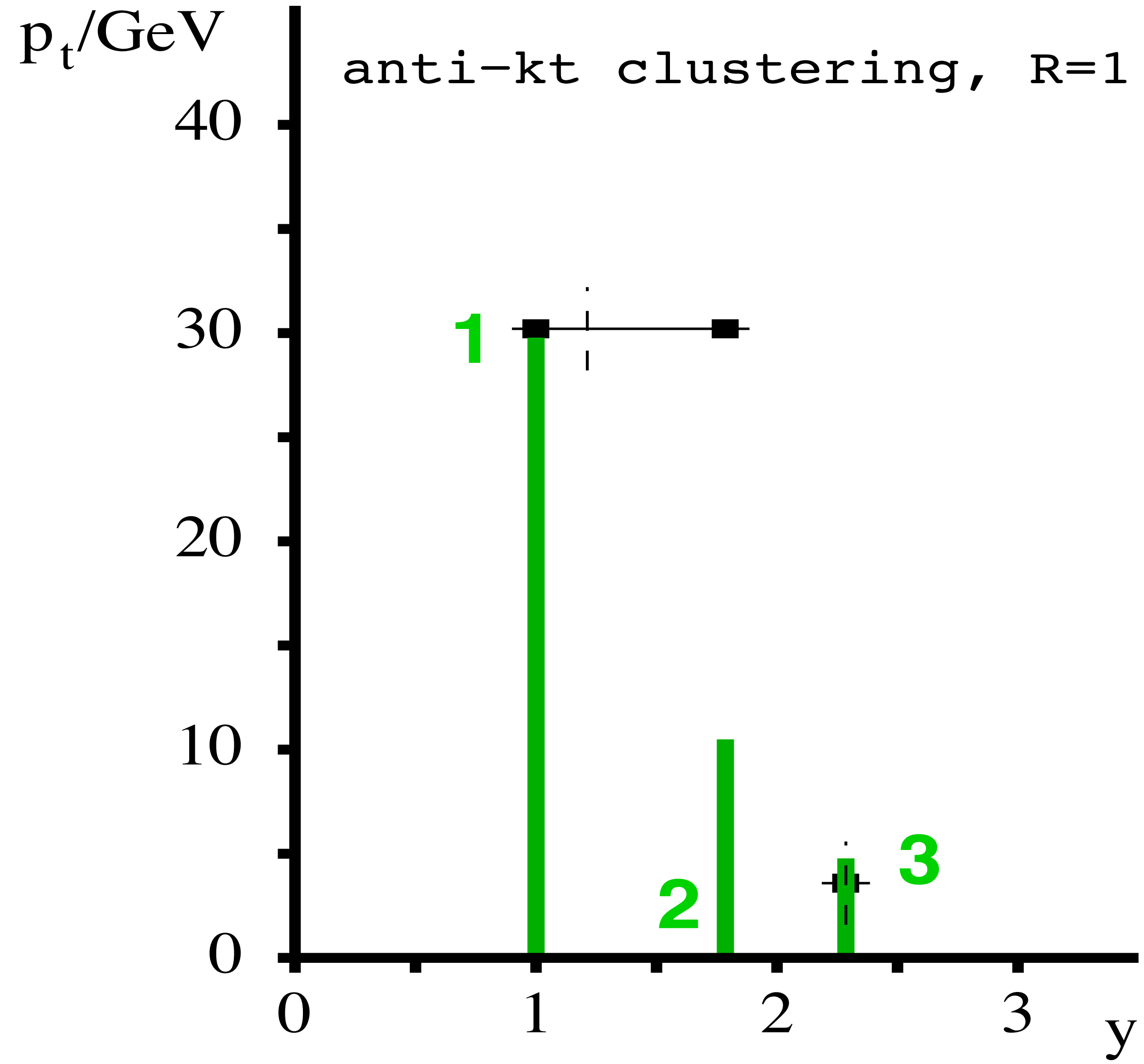


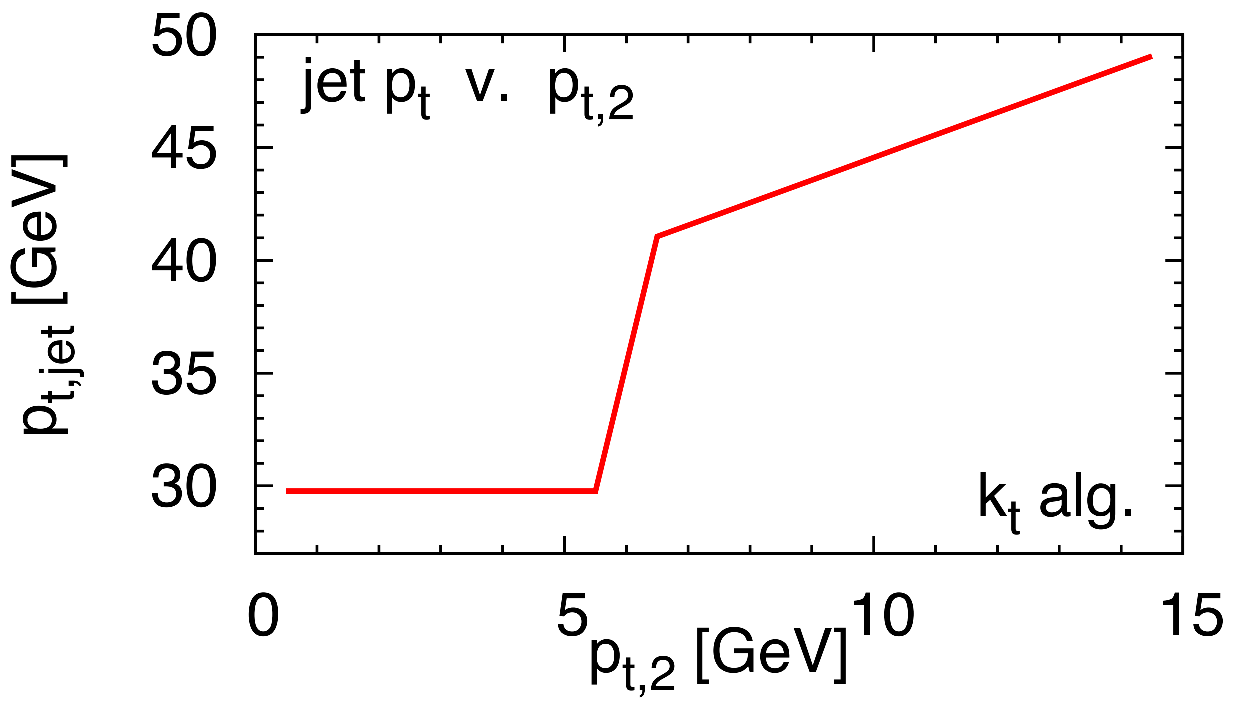
Linearity: k_t v. anti-k_t



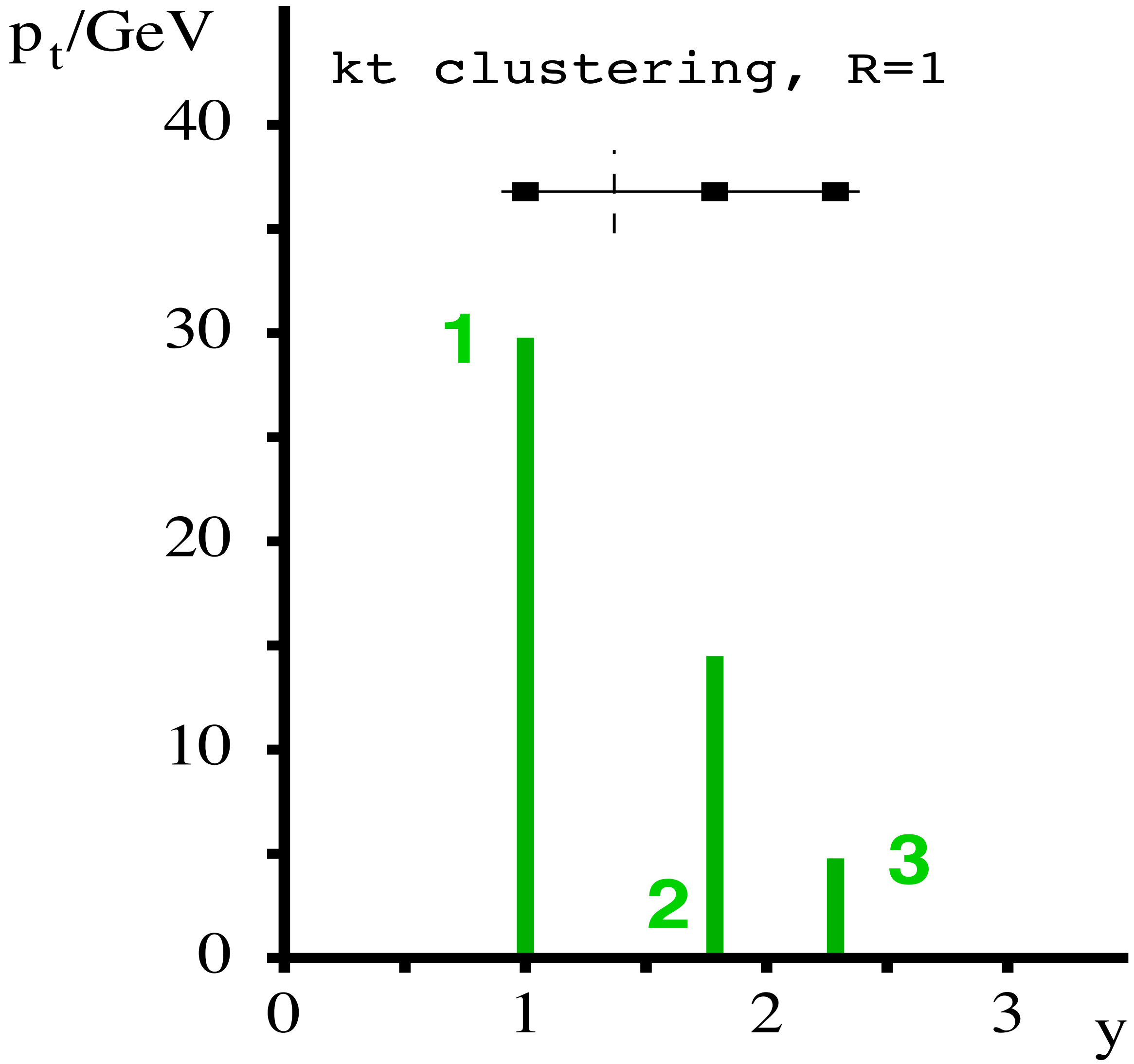
Linearity: k_t v. anti-k_t

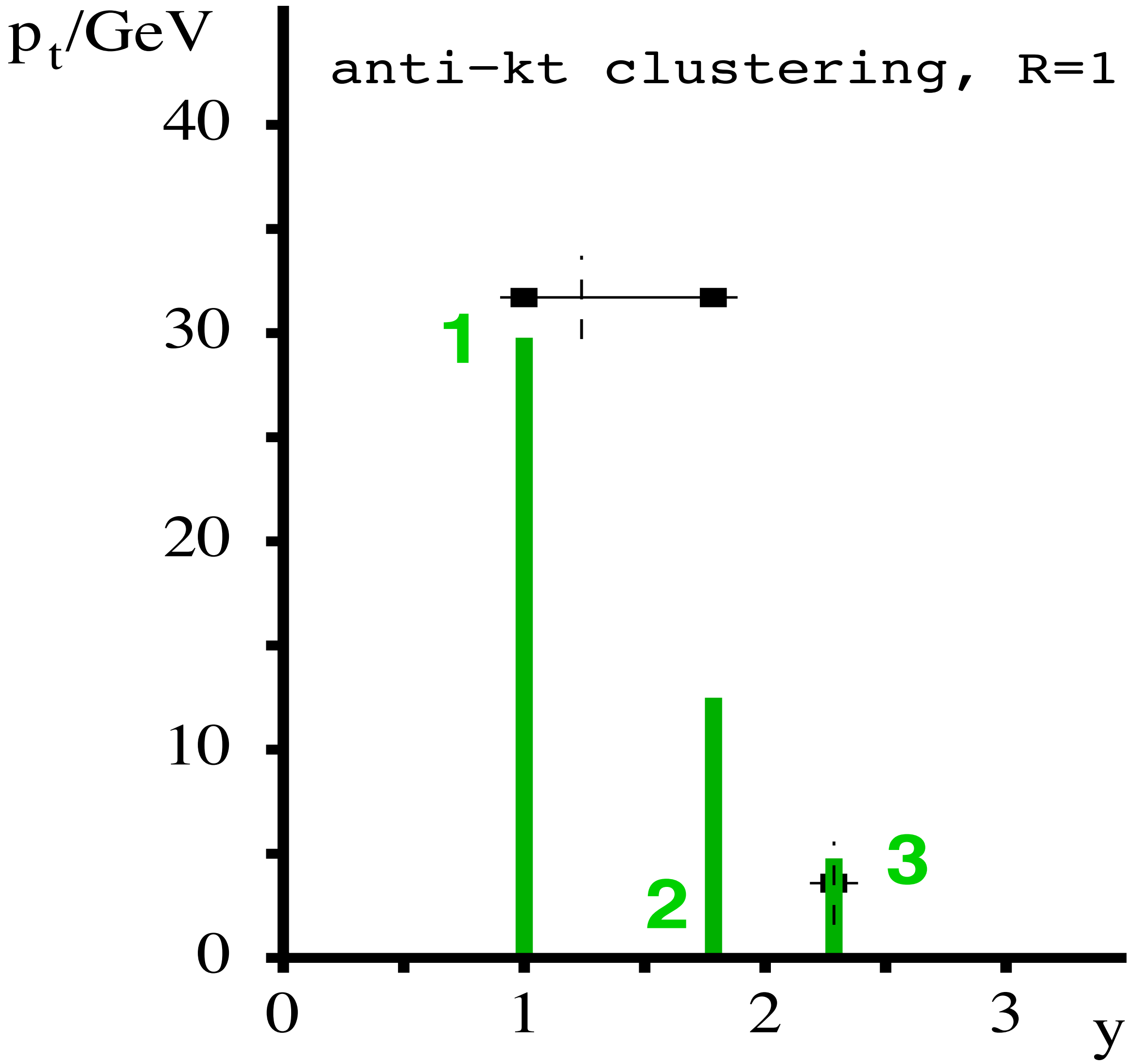


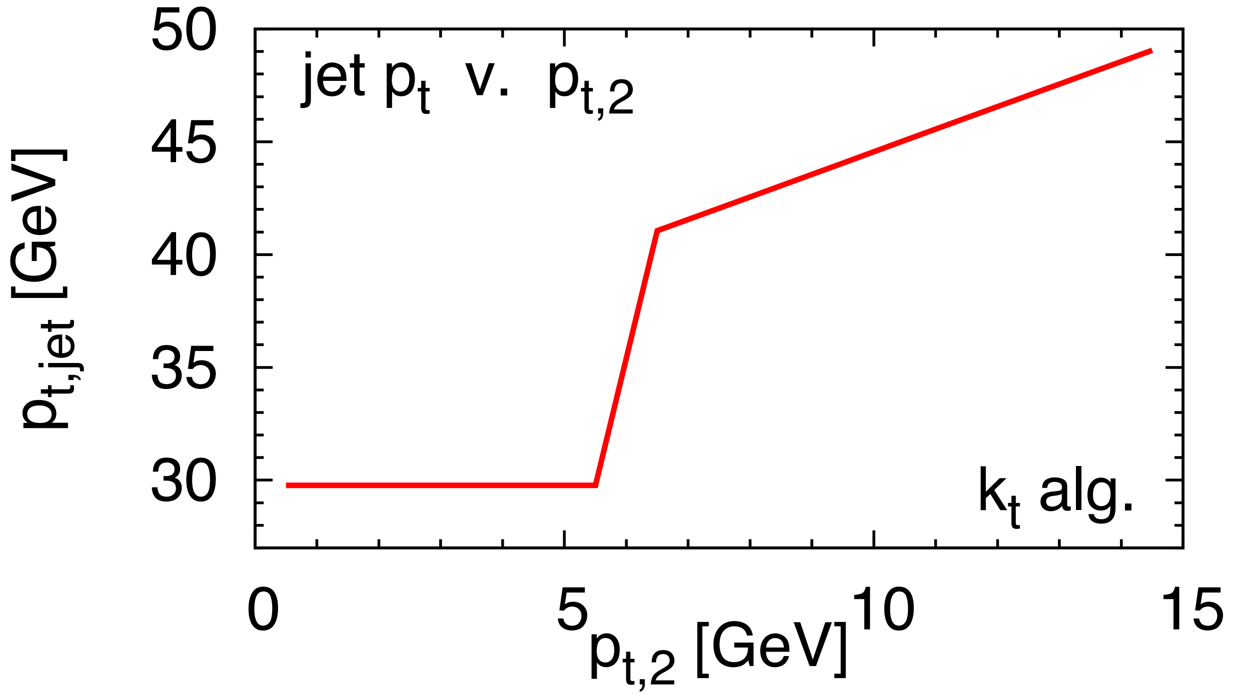




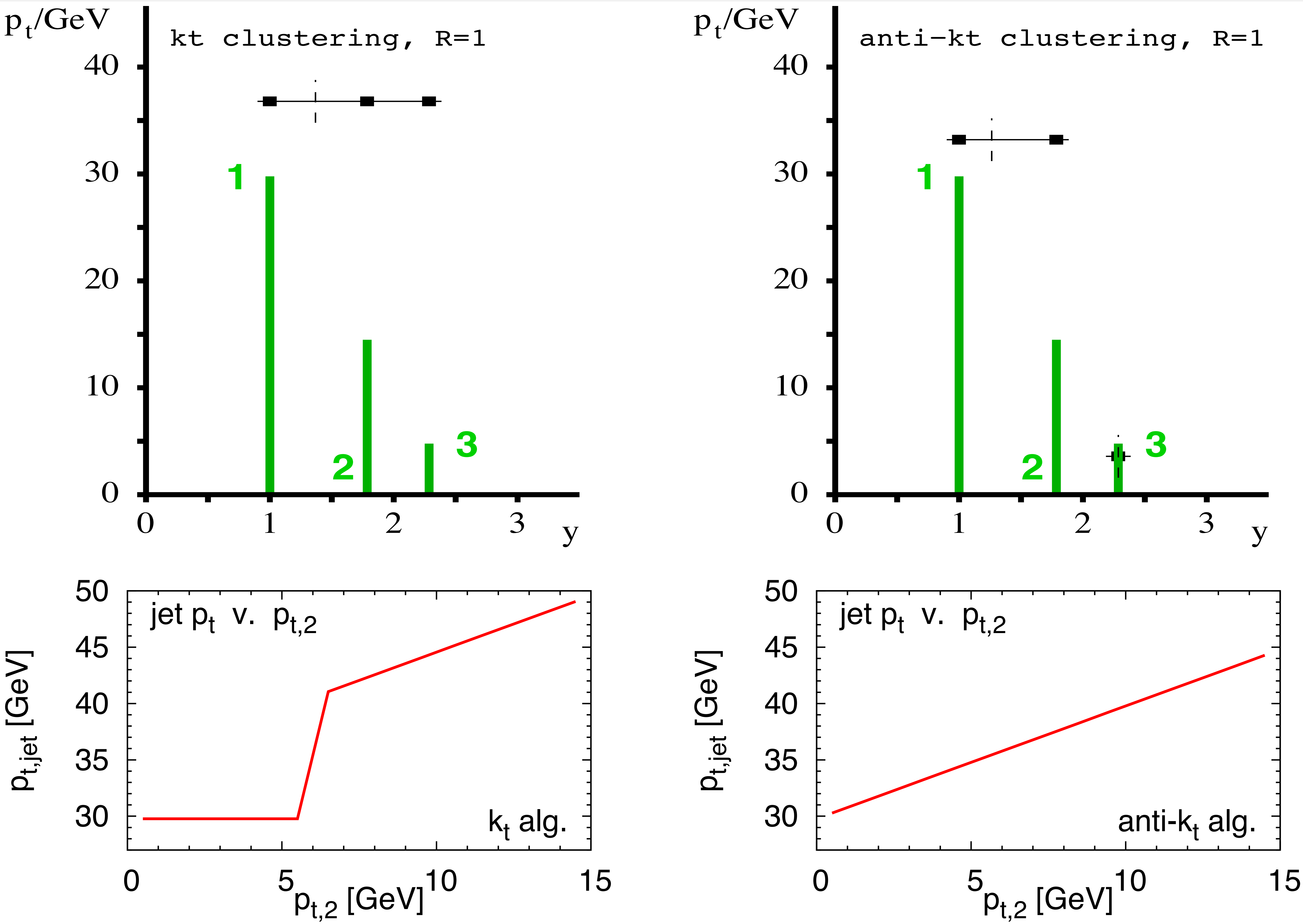
Linearity: k_t v. anti-k_t

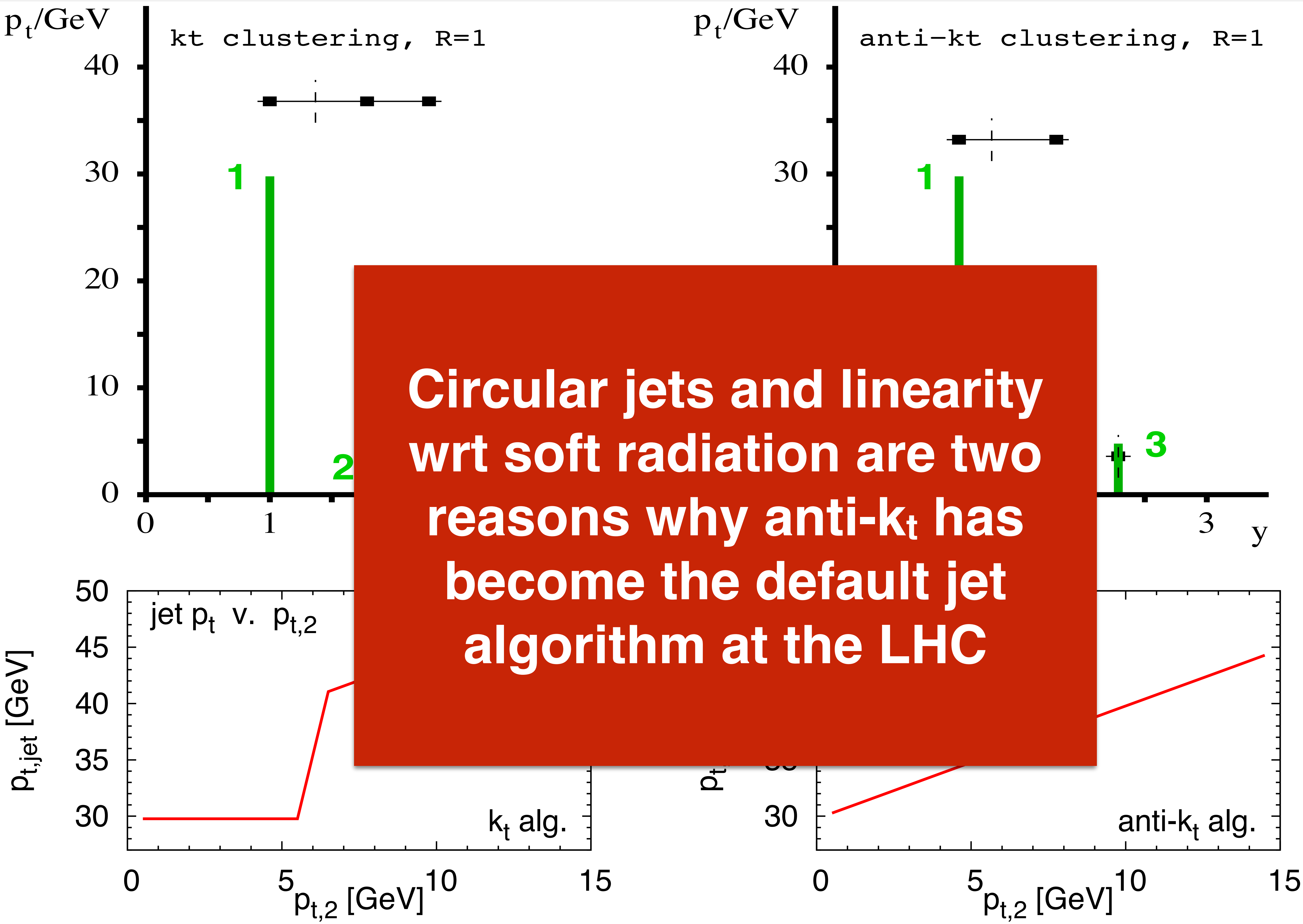






Linearity: k_t v. anti-k_t





Circular jets and linearity wrt soft radiation are two reasons why anti-k_t has become the default jet algorithm at the LHC

```
// specify a jet definition
double R = 0.4
JetDefinition jet_def(antikt_algorithm, R);
```

jet_algorithm can be any one of the four IRC safe pp-collider algorithms, or also a variety of e^+e^- algorithms, both native and plugins

```
// specify the input particles
vector<PseudoJet> input_particles = . . .;
```

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```

jet_algorithm can be any one of the four IRC safe pp-collider algorithms, or also a variety of e^+e^- algorithms, both native and plugins

```
// specify the input particles
vector<PseudoJet> input_particles = . . .;
```

```
// extract the jets
vector<PseudoJet> jets = jet_def(input_particles);

// pt of hardest jet
double pt_hardest = jets[0].pt();

// constituents of hardest jet
vector<PseudoJet> constituents = jets[0].constituents();
```

hadron collider jet reconstruction parameters

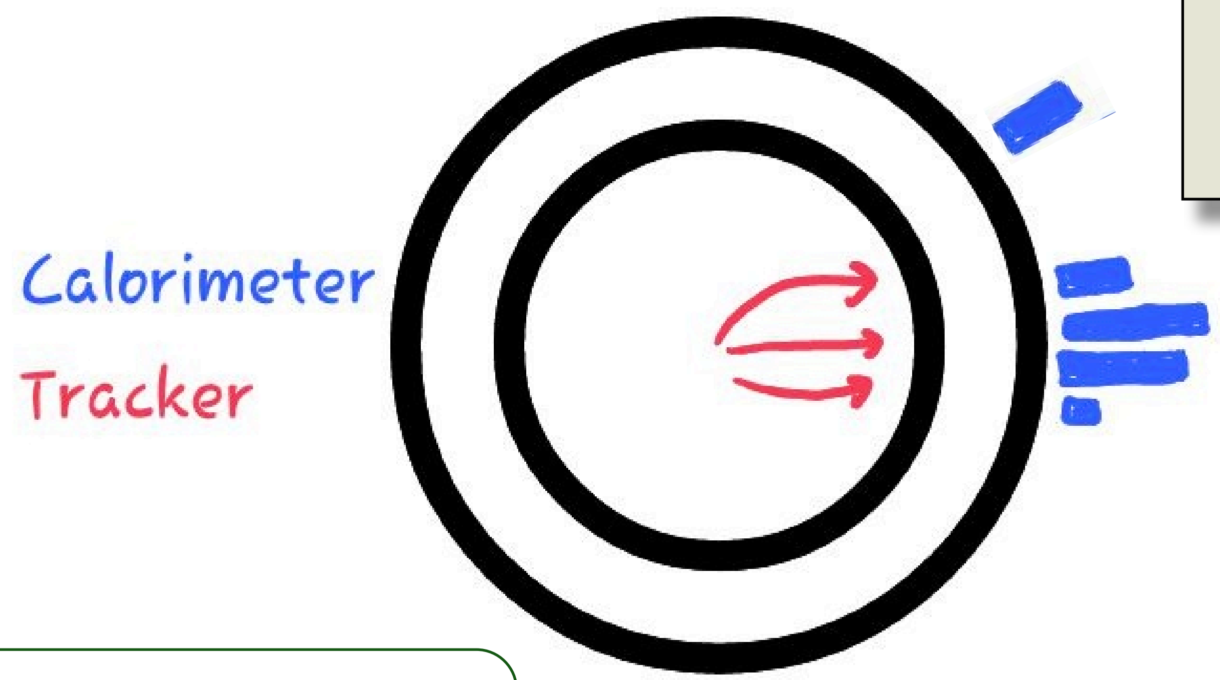
What goes into the jets?

ATLAS
Calorimeter towers, → increasingly replaced with **Particle/Unified Flow Objects**

CMS
"**Particle flow**" objects ~

1) charged tracks

2) neutrals: calorimeter towers not associated with charged tracks
(or leftover bits of calo if $E_{calo} - E_{track} \gg \sqrt{E_{calo}}$)



A particle-flow expert would shudder at this description

ideal detector image by Matthew Low

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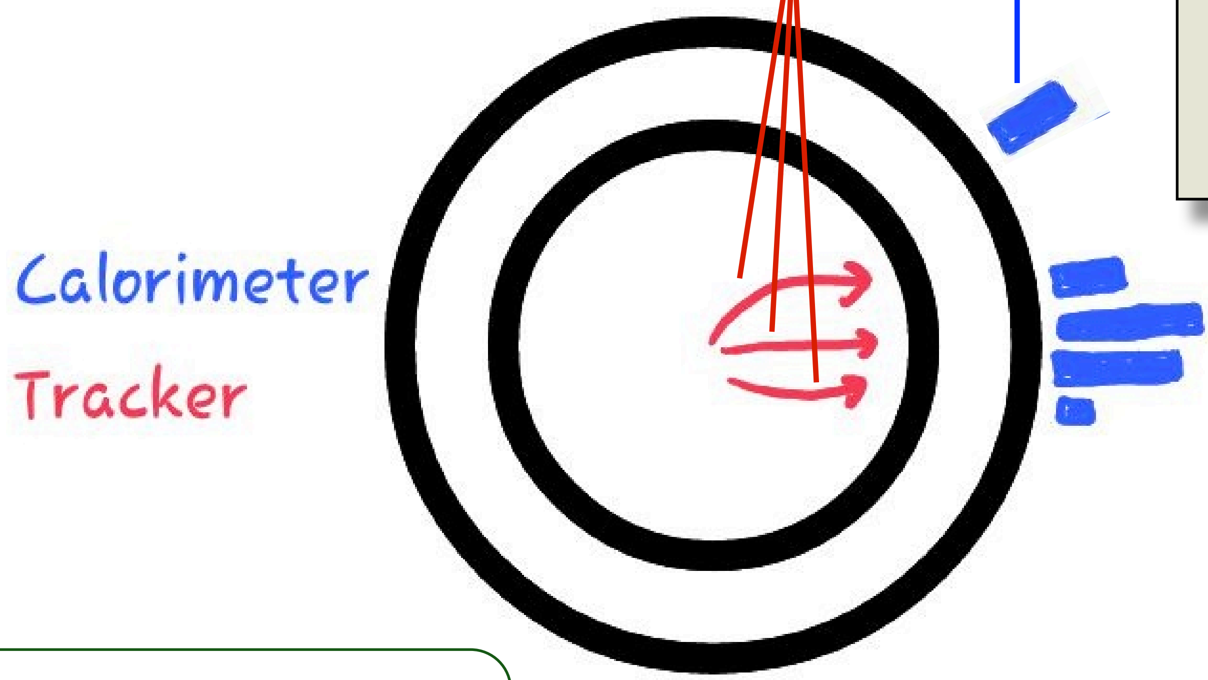

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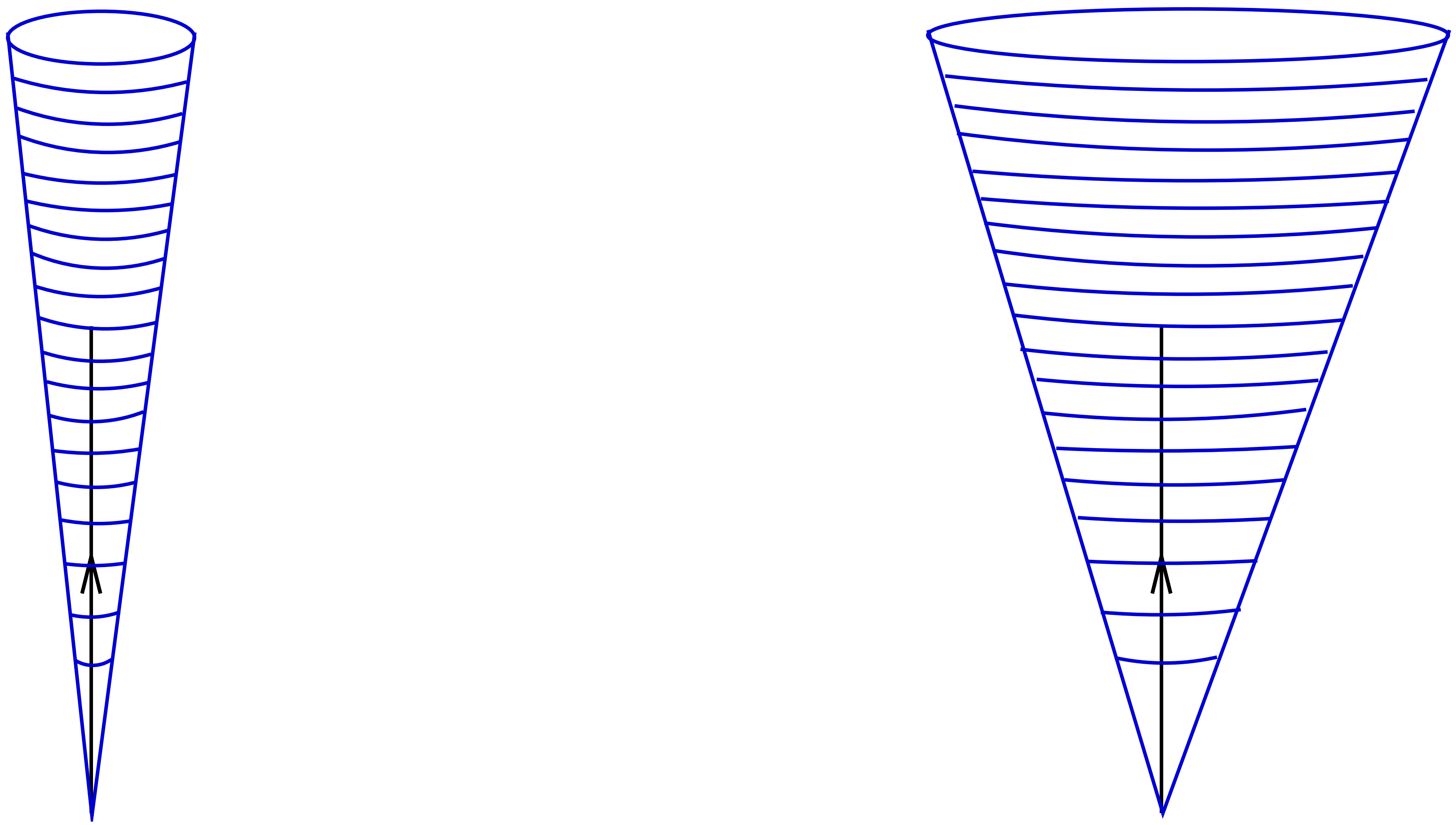
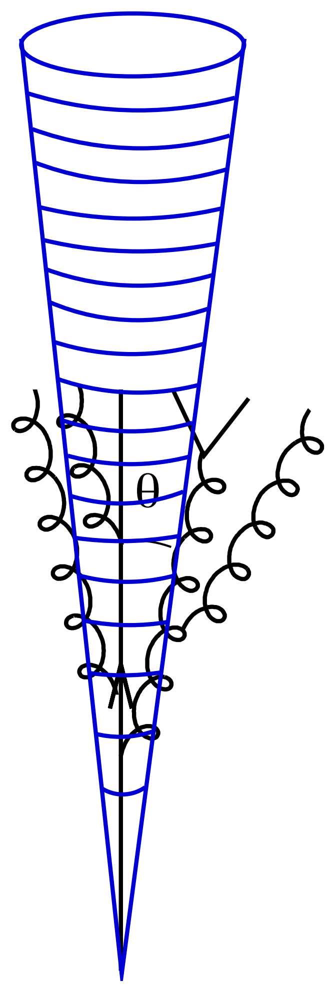
ideal detector image by Matthew Low

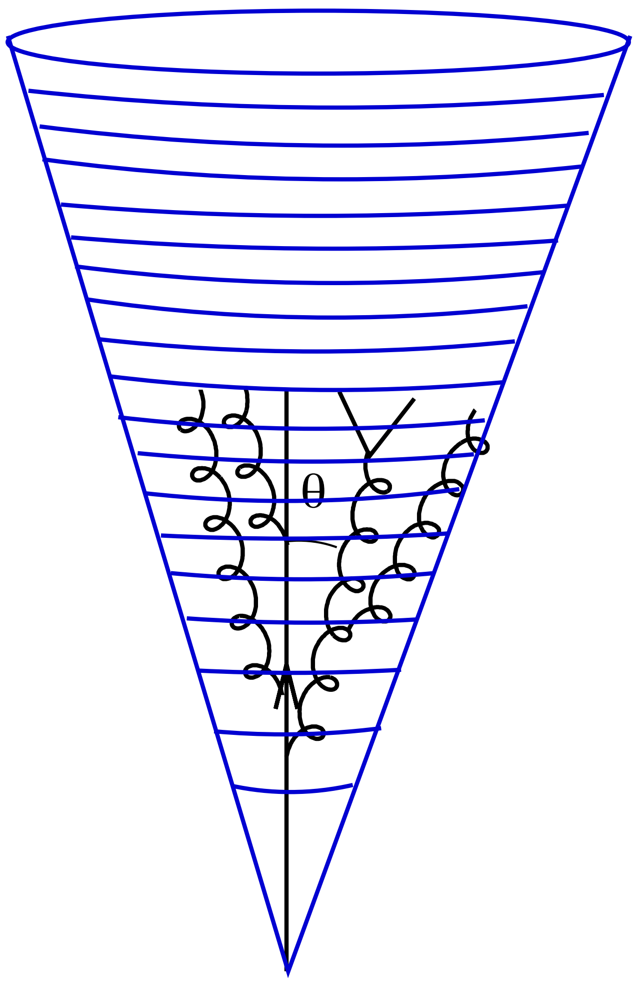
Small jet radius **Large jet radius**



single parton @ LO: **jet radius irrelevant**

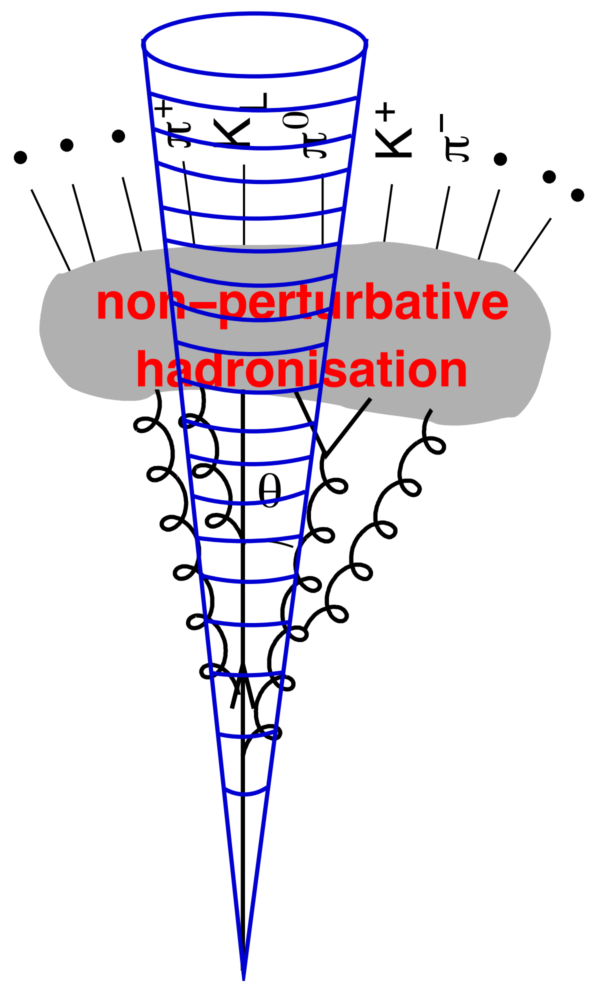
Small jet radius **Large jet radius**



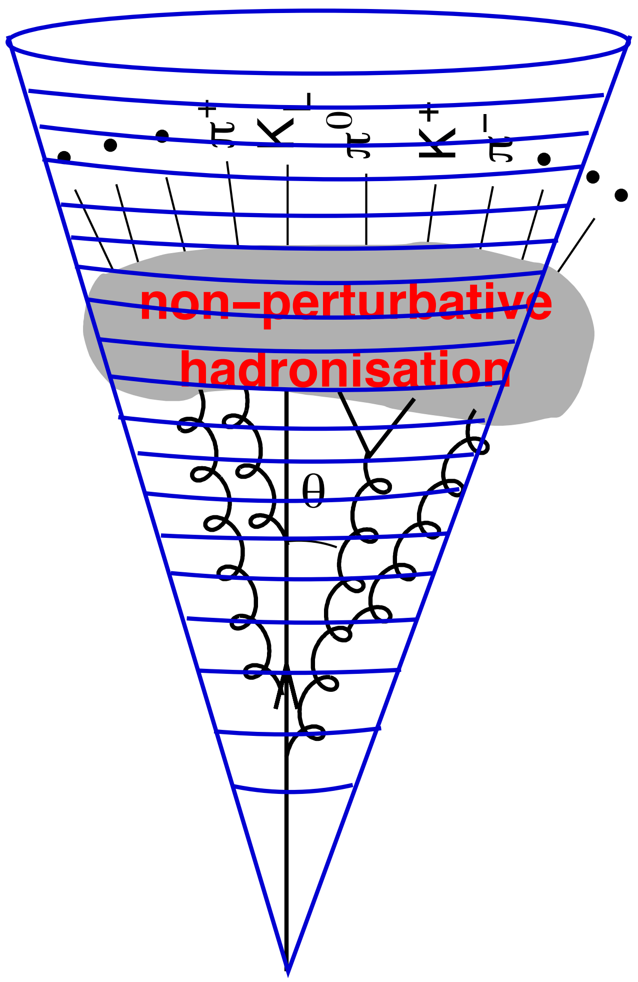


perturbative fragmentation: **large jet radius better**
(it captures more)

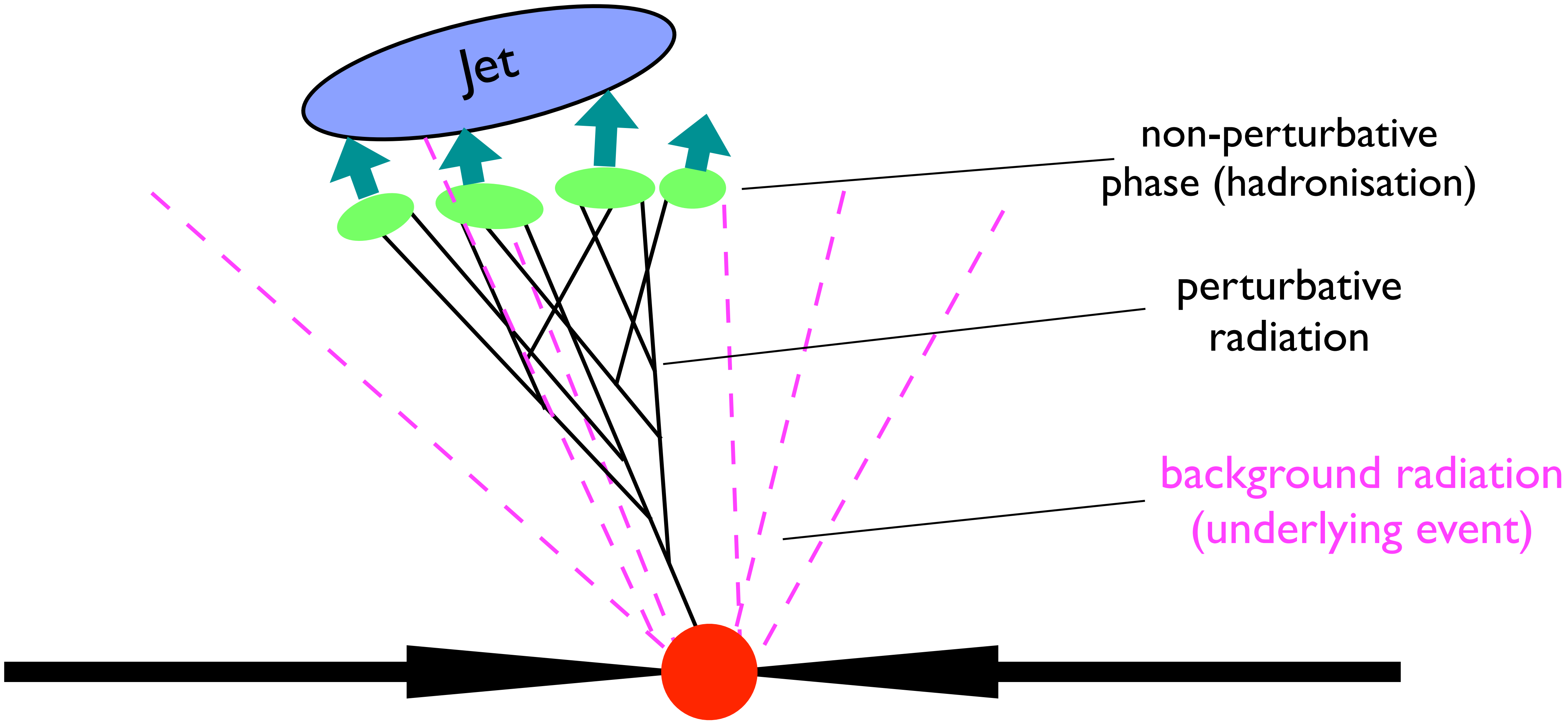
Small jet radius

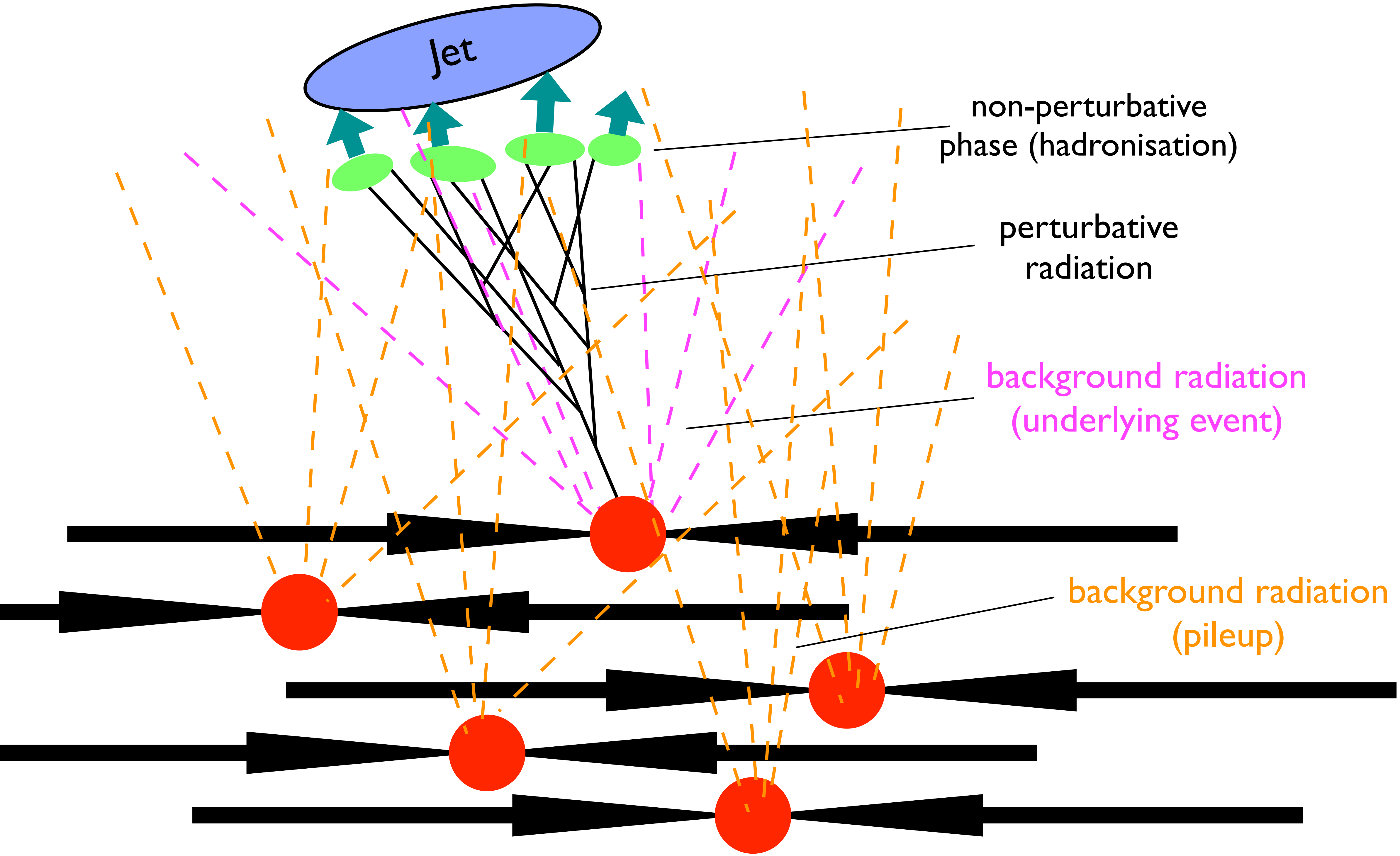


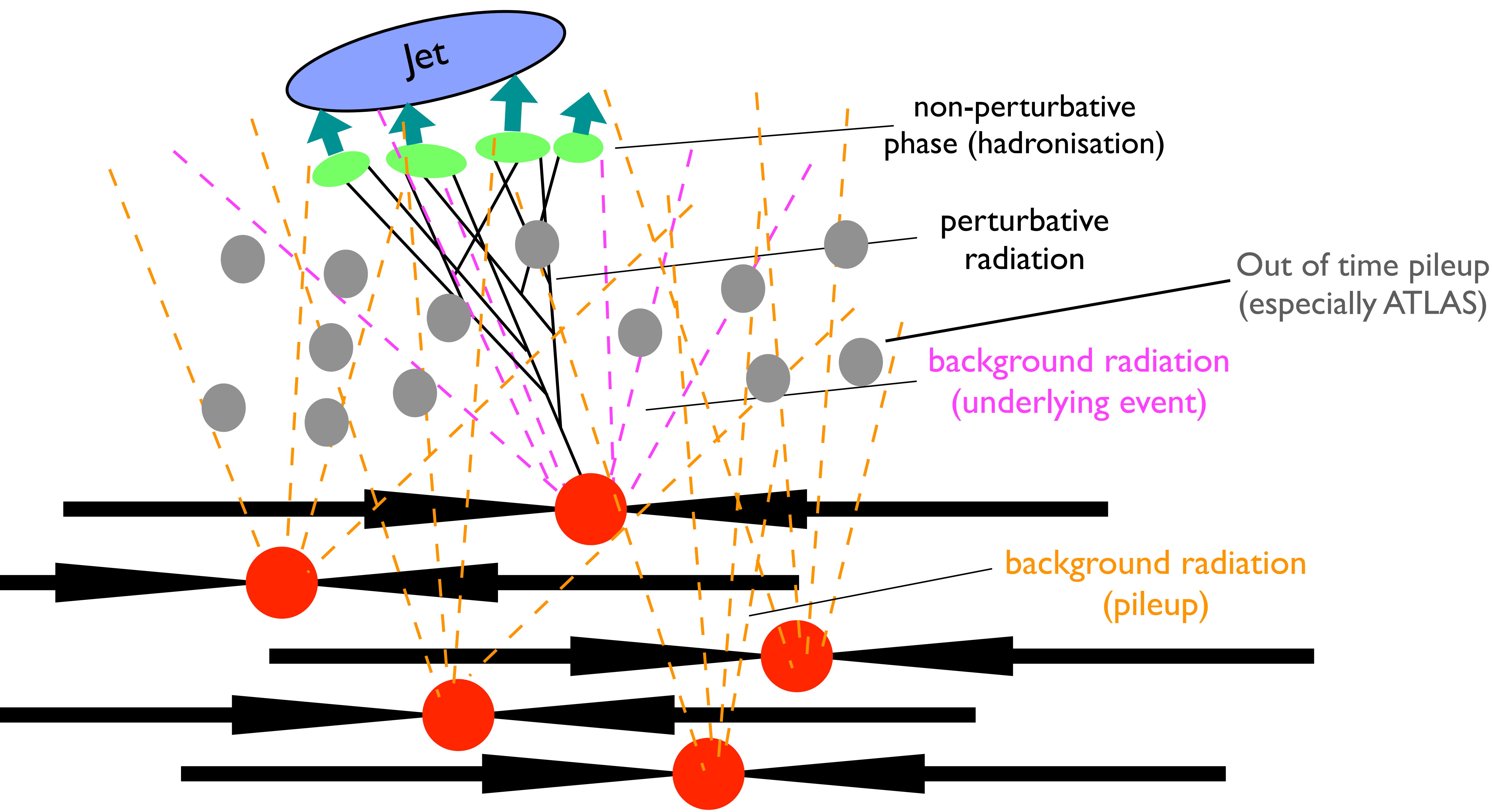
Large jet radius



non-perturbative fragmentation: **large jet radius better**
(it captures more)

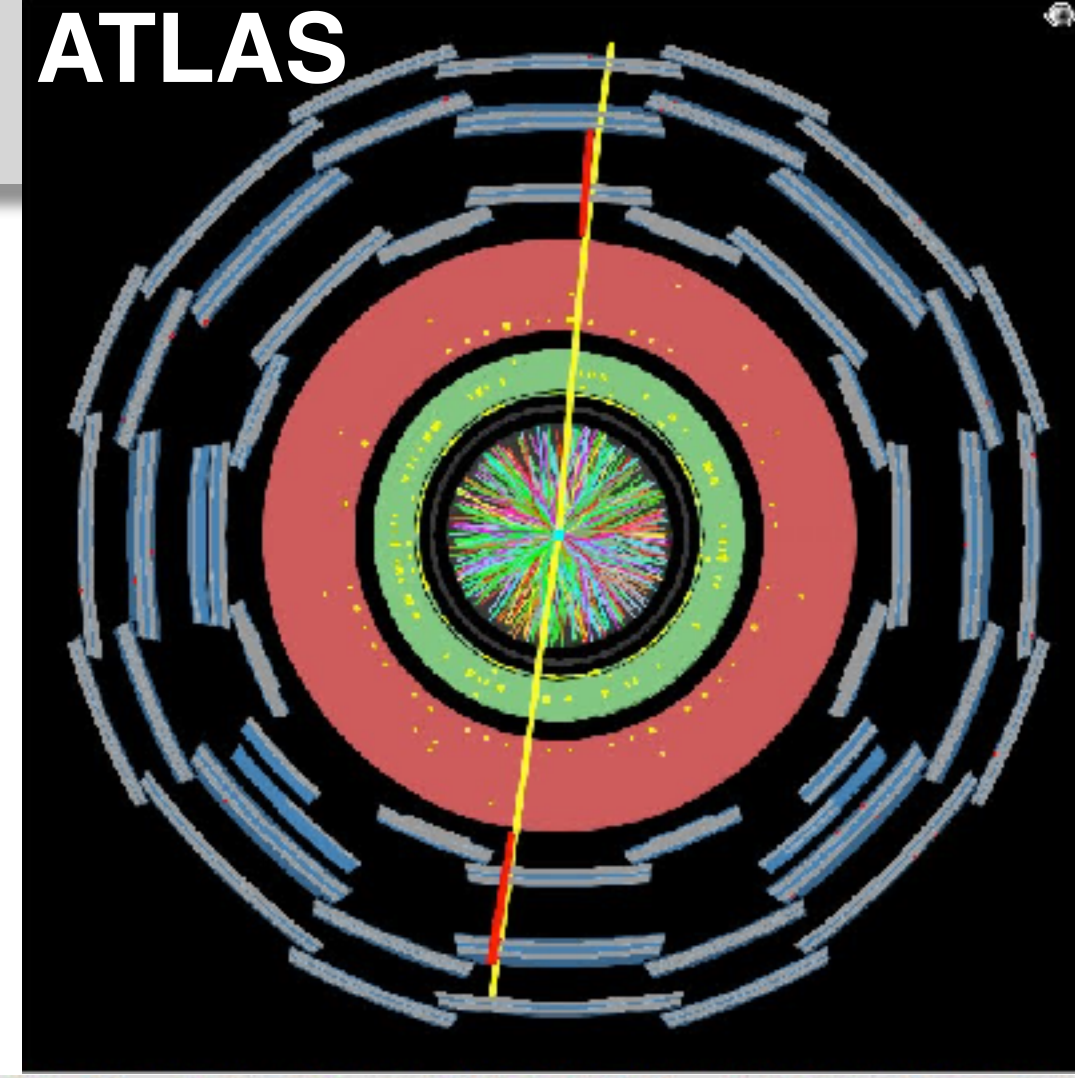

Jet
non-perturbative
phase (hadronisation)
perturbative
radiation
background radiation
(underlying event)


Jet
non-perturbative
phase (hadronisation)
perturbative
radiation
background radiation
(underlying event)
background radiation
(pileup)

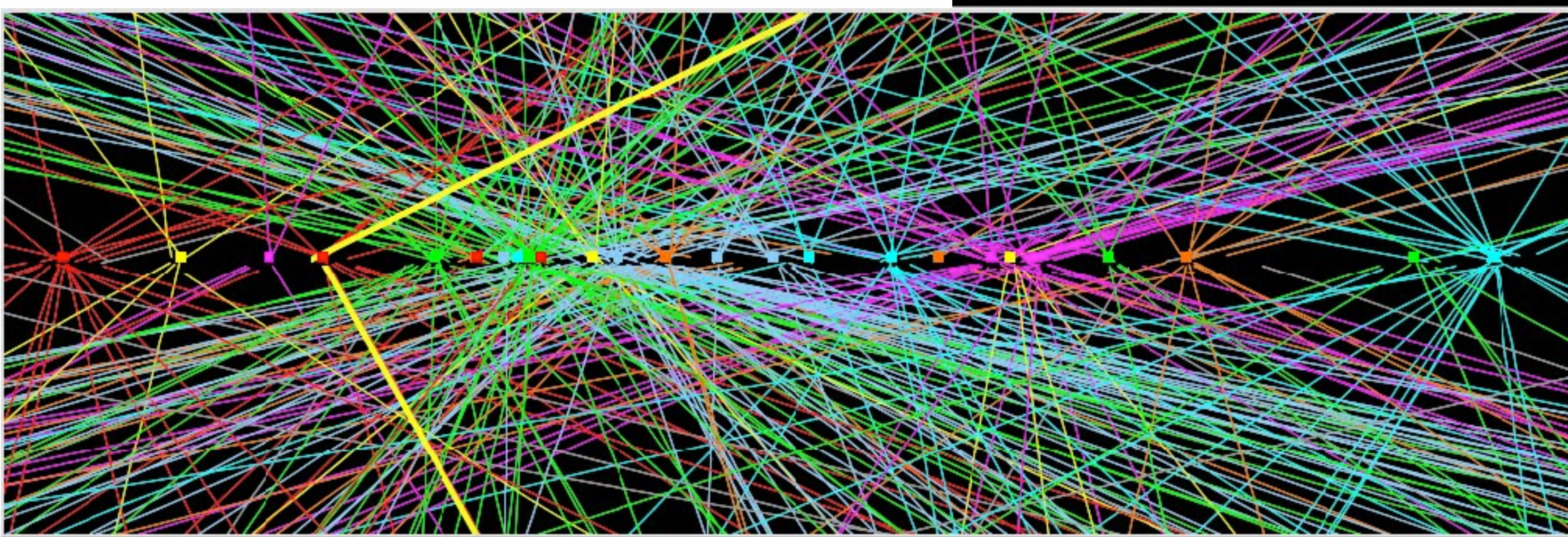

Jet
non-perturbative
phase (hadronisation)
perturbative
radiation
Out of time pileup
(especially ATLAS)
background radiation
(underlying event)
background radiation
(pileup)

Pileup for real

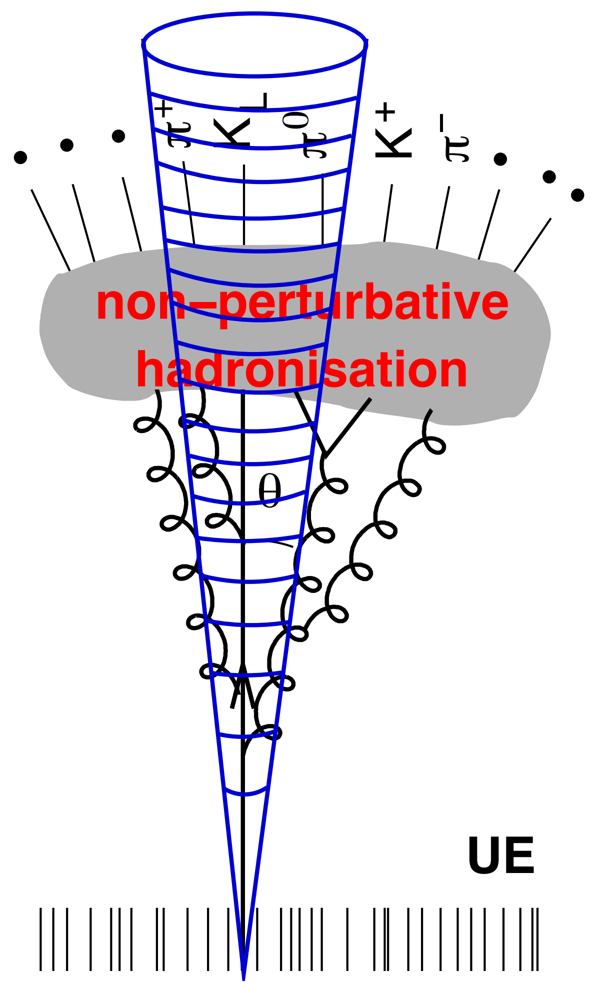
~ 20 m



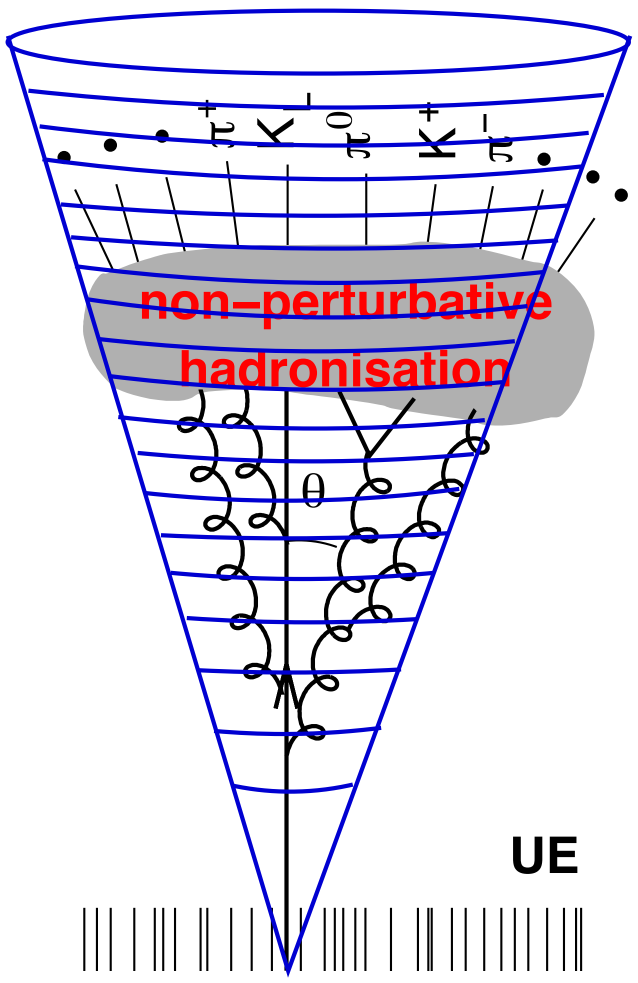
a few cm



Small jet radius

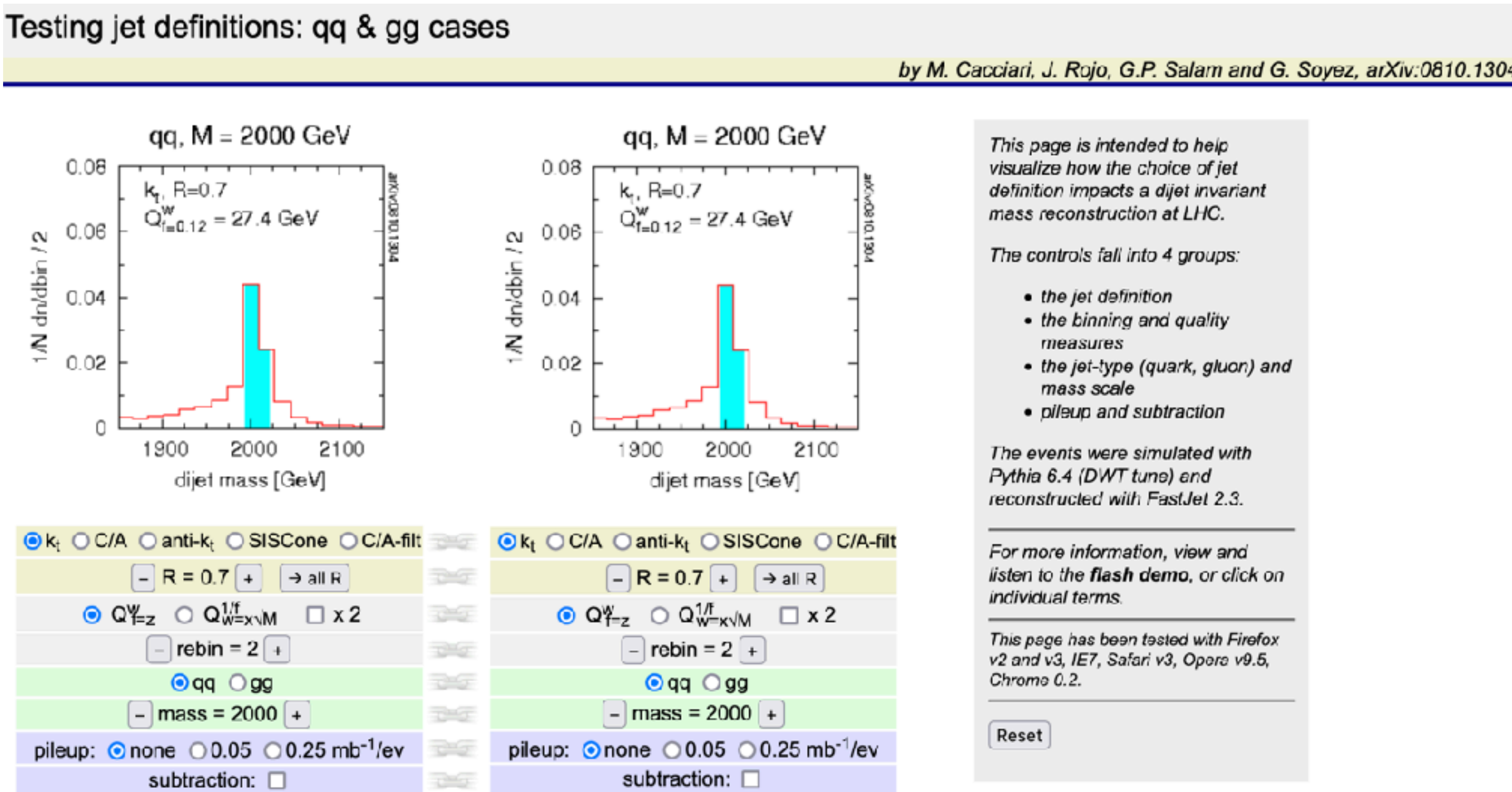


Large jet radius

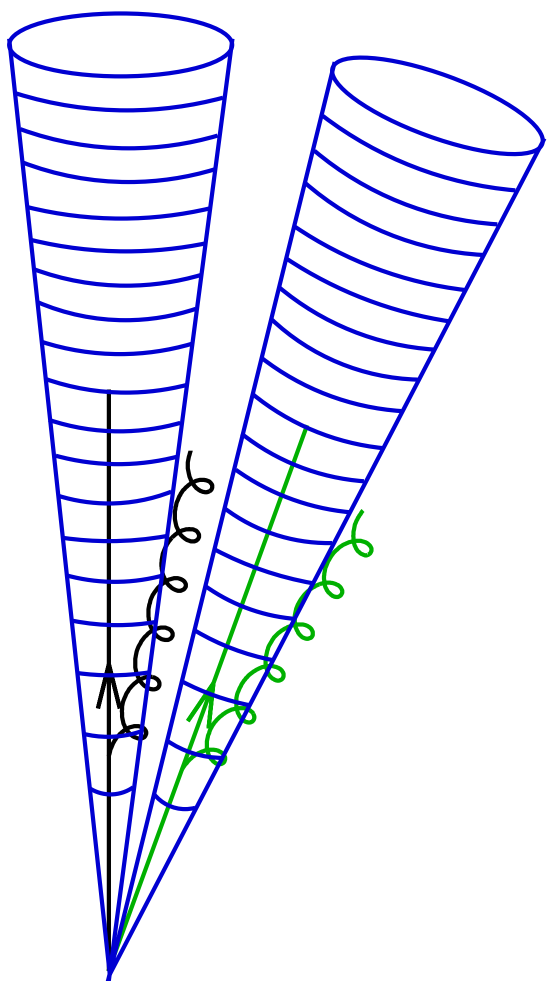


underlying ev. & pileup "noise": **small jet radius better**
(it captures less)

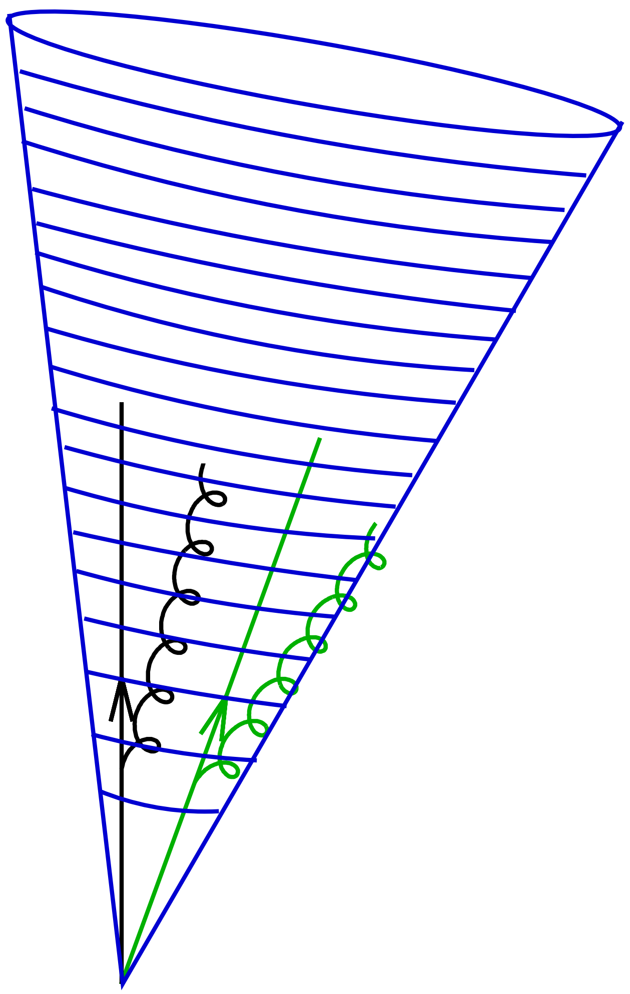
https://www.lpthe.jussieu.fr/~salam/jet-quality/



Small jet radius



Large jet radius



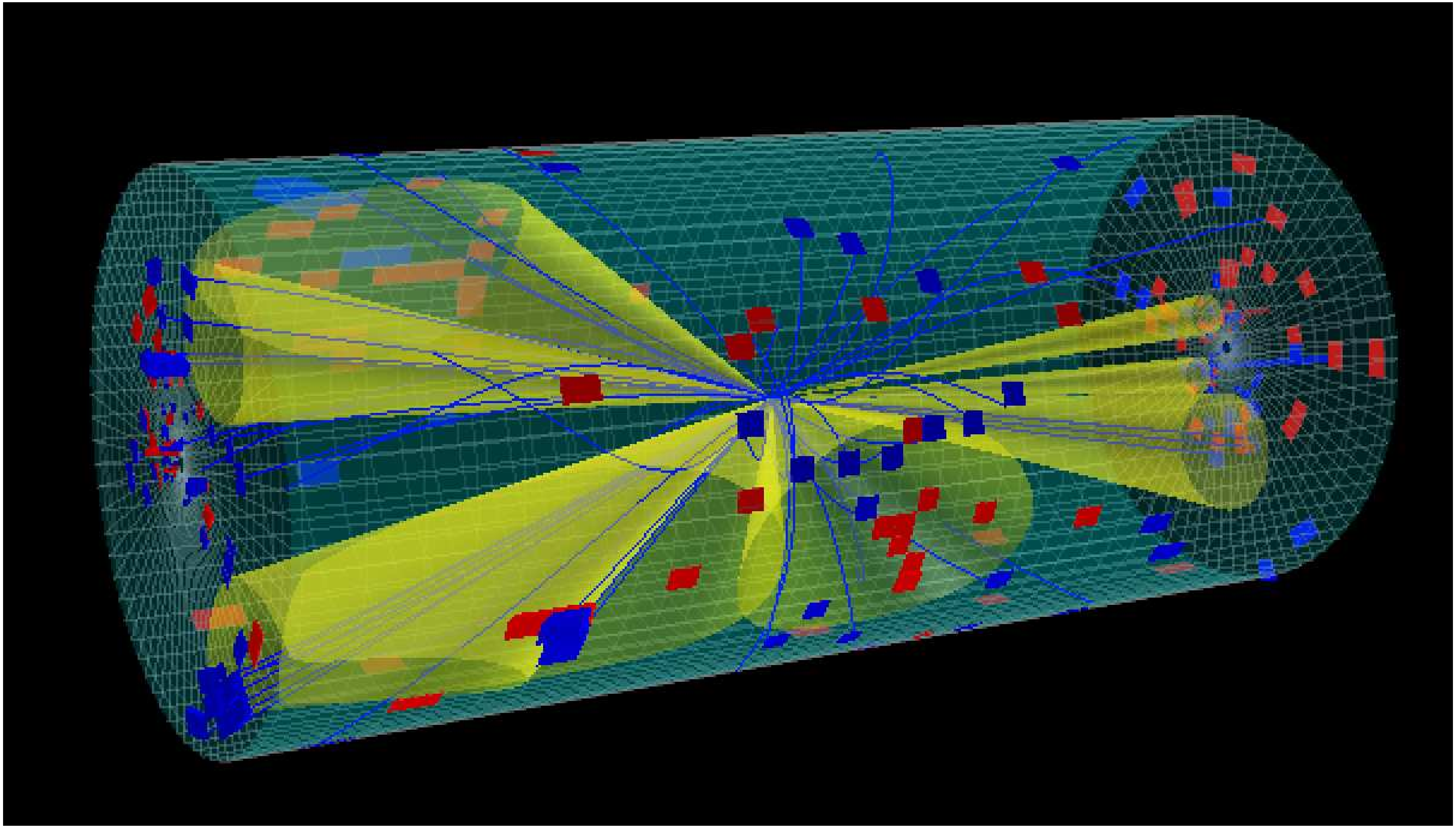
multi-hard-parton events: **small jet radius better**
(it resolves partons more effectively)

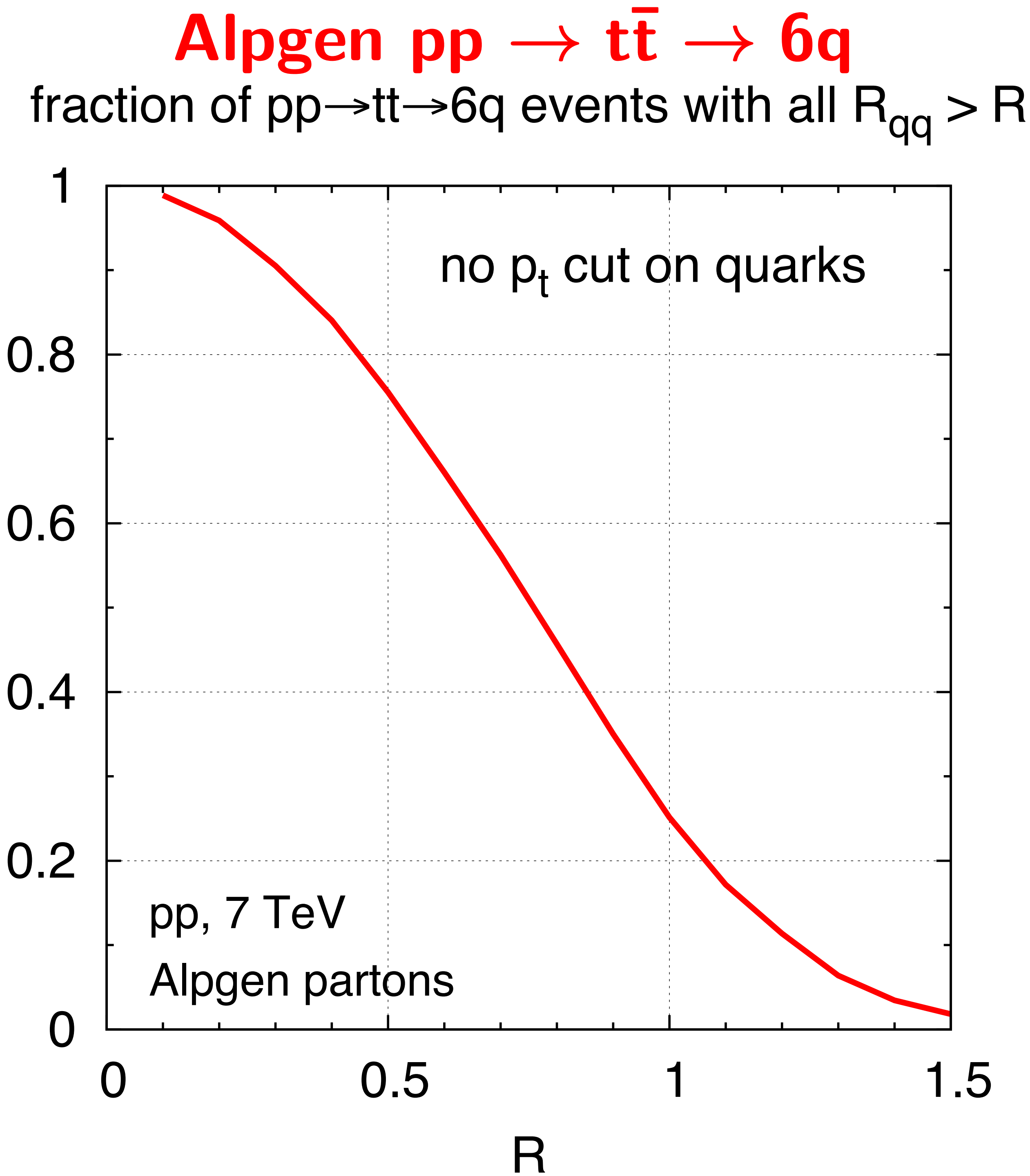
Can we capture all quarks and gluons?

Should we capture all quarks and gluons?

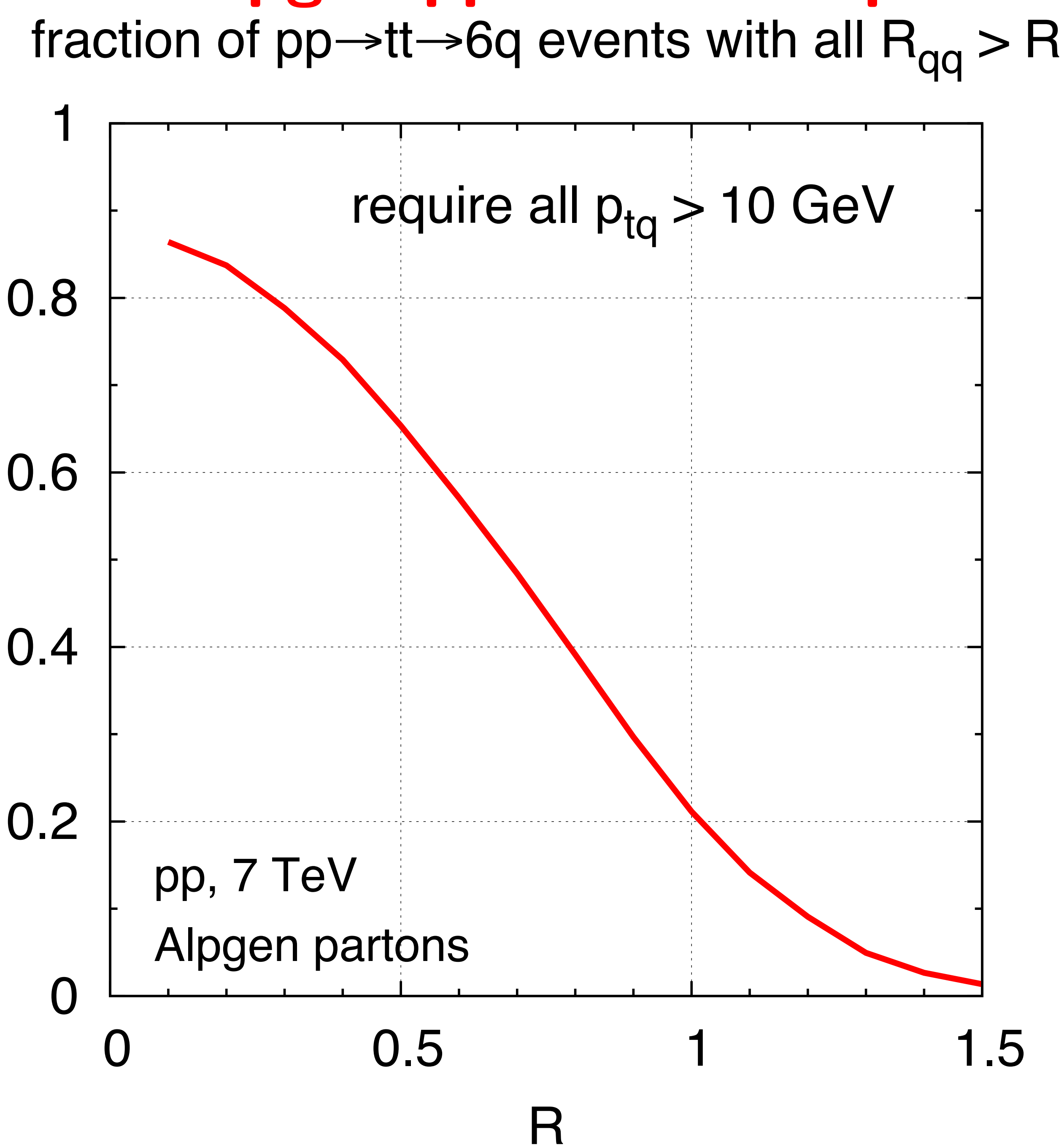
$pp \to t\bar{t}$

simulated with Pythia, displayed with Delphes

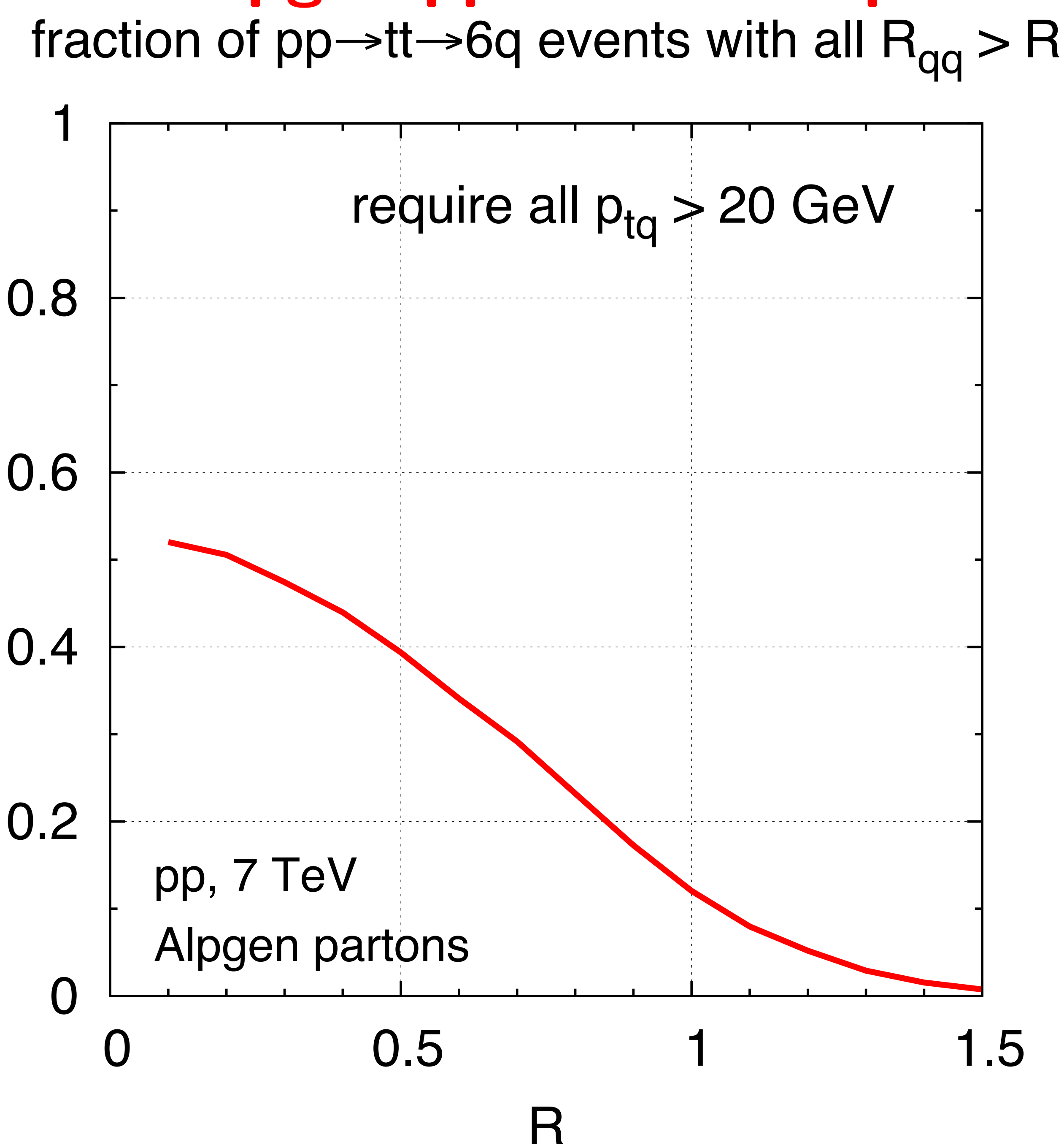


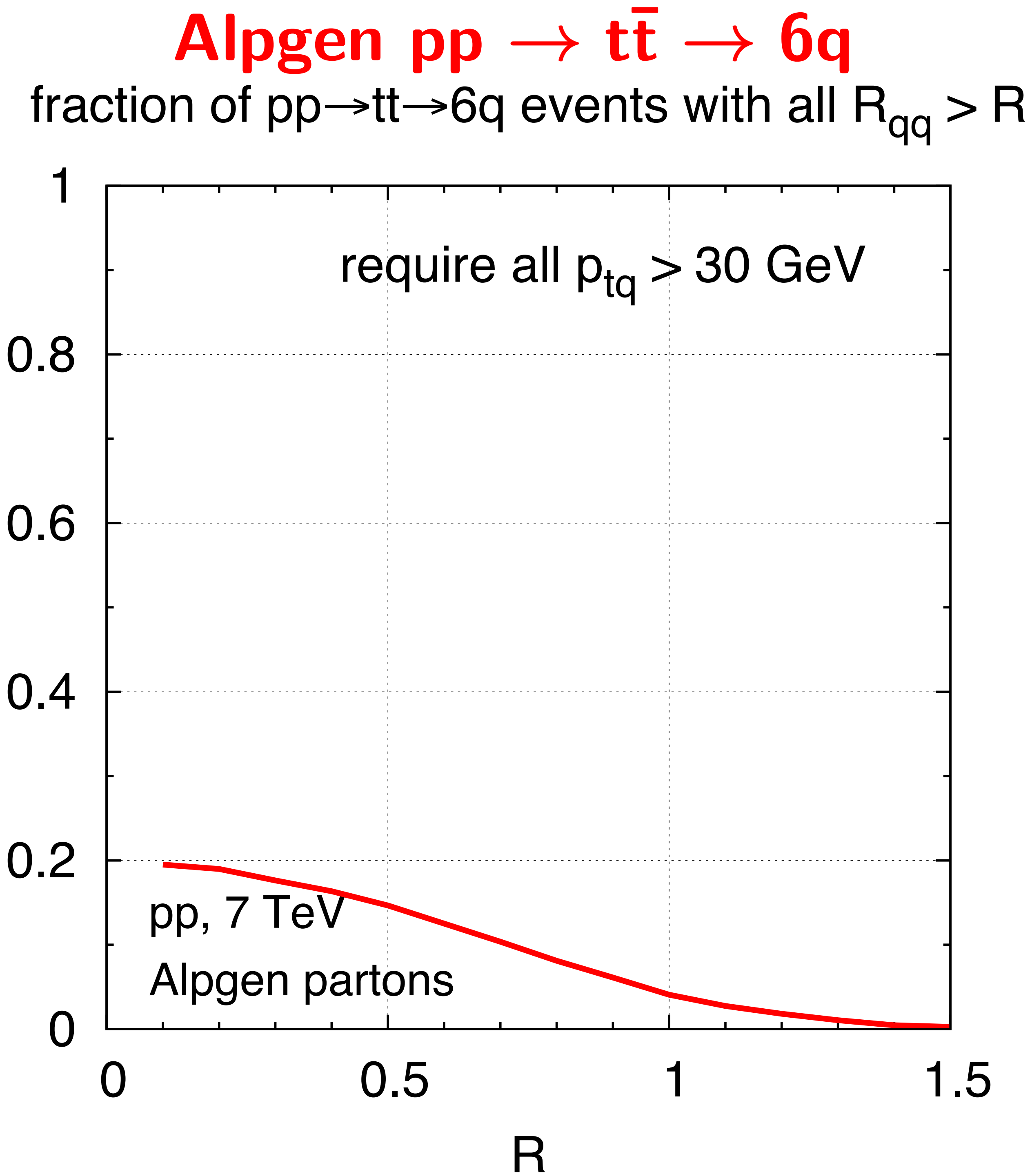

Alpgen pp → $t\bar{t}$ → 6q
fraction of pp→tt→6q events with all R_{qq} > R
no p_t cut on quarks
pp, 7 TeV
Alpgen partons
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
R

Alpgen $pp \to t\bar{t} \to 6q$

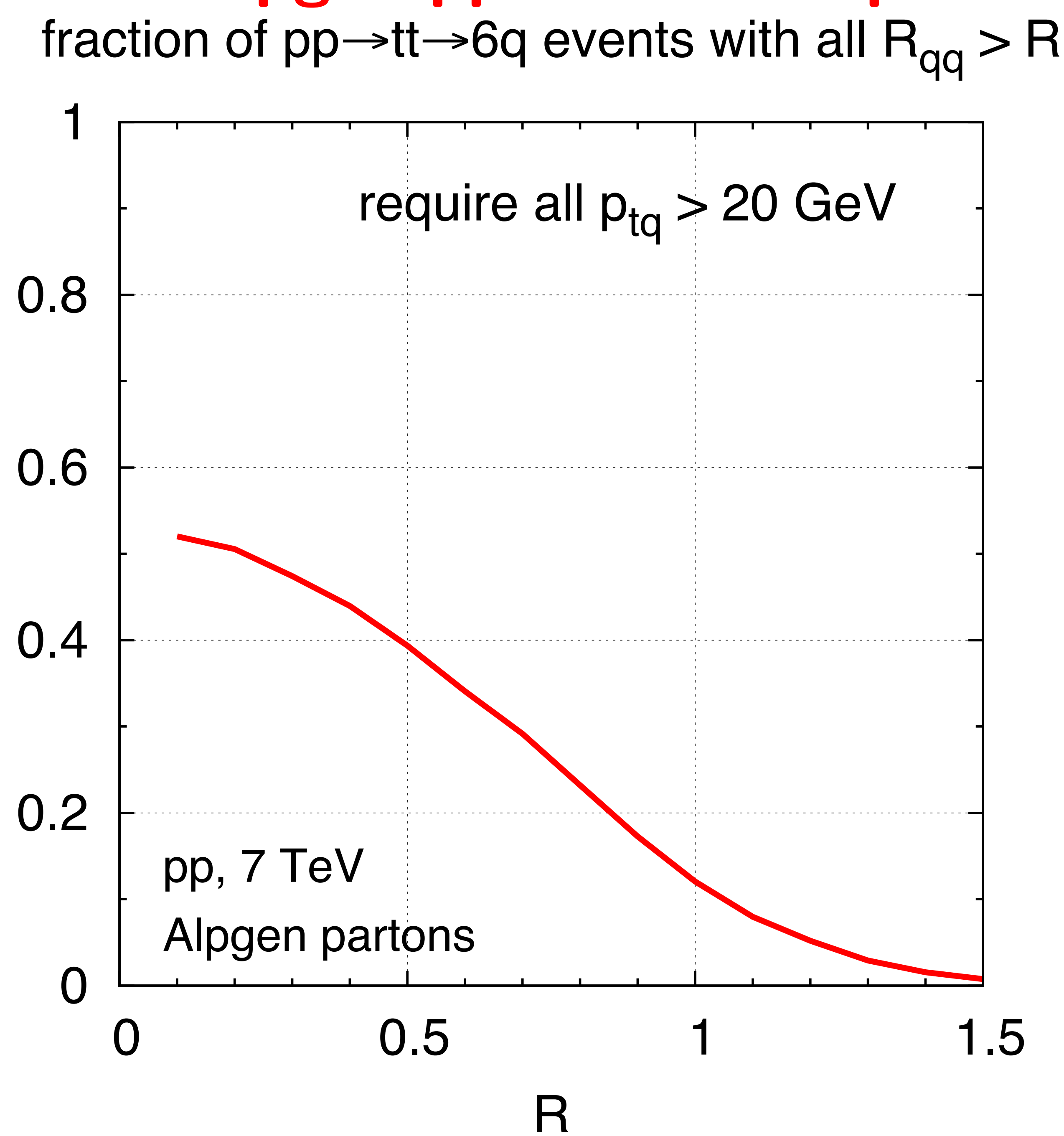


Alpgen $pp \to t\bar{t} \to 6q$



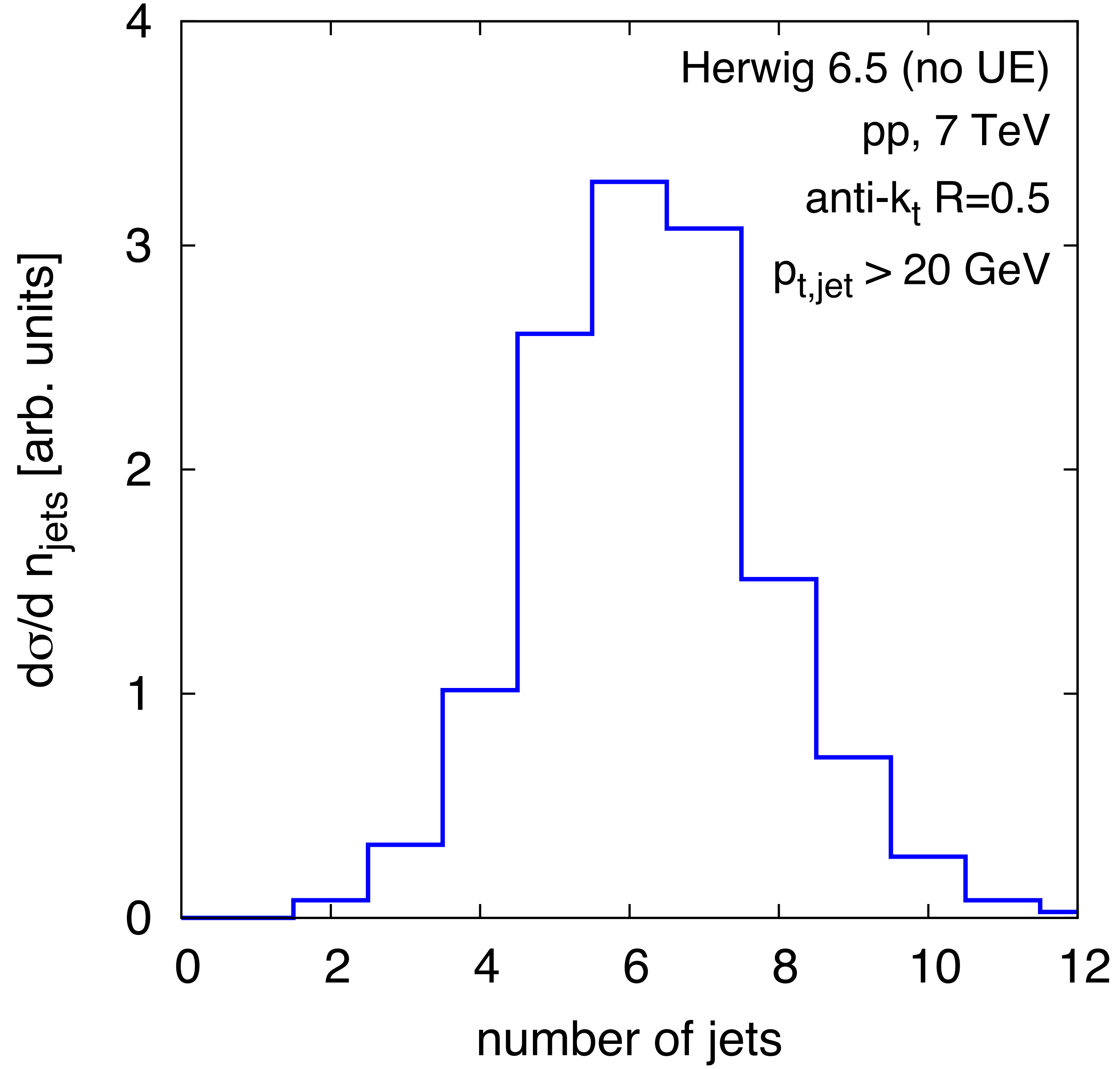

Alpgen pp → tt̄ → 6q
fraction of pp→tt→6q events with all R_qq > R
require all p_tq > 30 GeV
pp, 7 TeV
Alpgen partons
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
R

Alpgen pp $\to t\bar{t} \to$ 6q



Herwig pp $\to t\bar{t} \to$ hadrons

Distribution of number of jets



Standard LHC jet finding

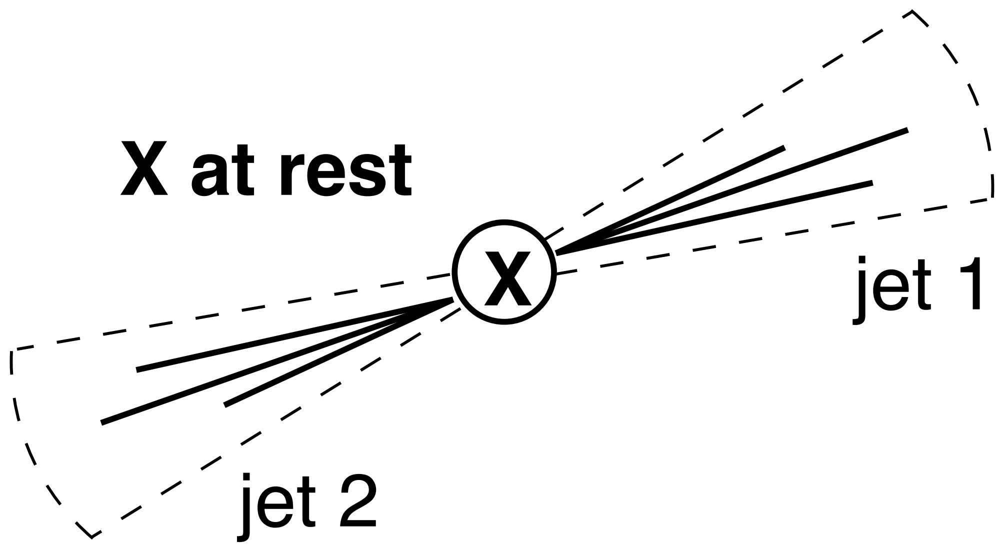
- Uses the anti-k_t algorithm
- Uses a jet radius R=0.4
- Uses a transverse momentum threshold that is typically at least 20 GeV (exact value depends on the analysis)

Radius and p_T threshold choices give a good compromise between

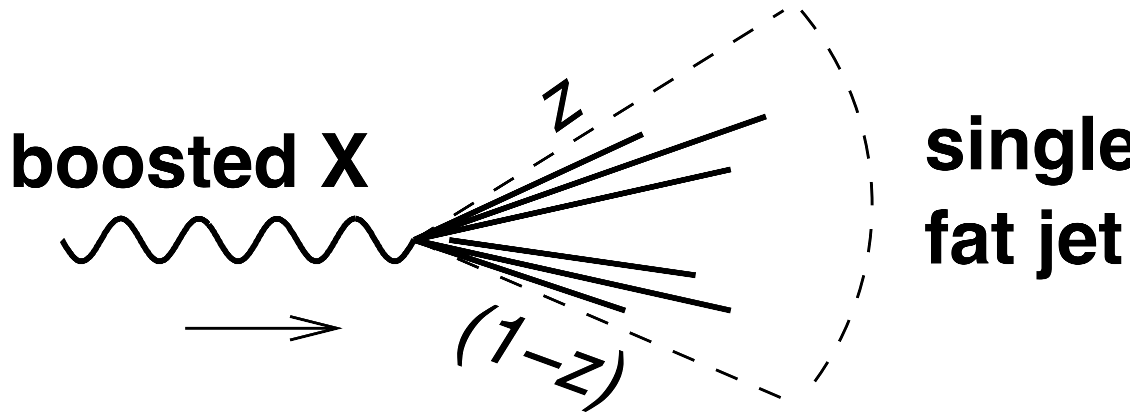
- ability to resolve multi-jet physics
- loss of radiation from jets
- additional spurious jets
- contamination from pileup

boosted object reconstruction

Normal analyses: two quarks from $X \to q\bar{q}$ reconstructed as two jets

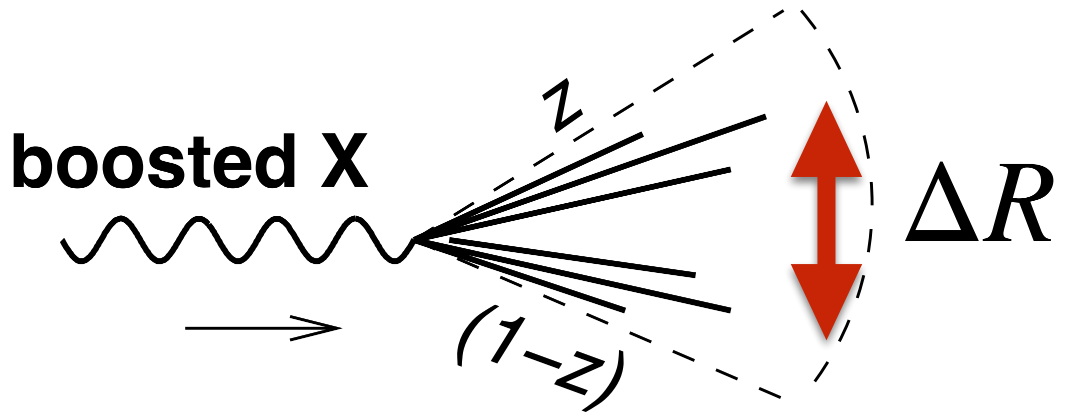


High-p_t regime: EW object X is boosted, decay is collimated, $q\bar{q}$ both in same jet



High-p_t regime: EW object X is boosted, decay is collimated, $q\bar{q}$ both in same jet

$$m_X^2 \simeq E_X^2 \cdot z(1-z) \cdot 2(1-\cos\theta)$$
$$\simeq p_{tX}^2 \cdot z(1-z) \cdot \Delta R^2$$



The two prongs end up in a single jet if

$$\Delta R \simeq \frac{m}{p_t}\frac{1}{\sqrt{z(1-z)}} \sim \frac{2m}{p_t} < R \quad \text{or} \quad p_t \gtrsim \frac{2m}{R}$$

E.g. W-boson with $p_t > 400\,\mathrm{GeV}$ ends up collimated in a single jet.

Two widely used terms
though there's not a
consensus about
what they mean

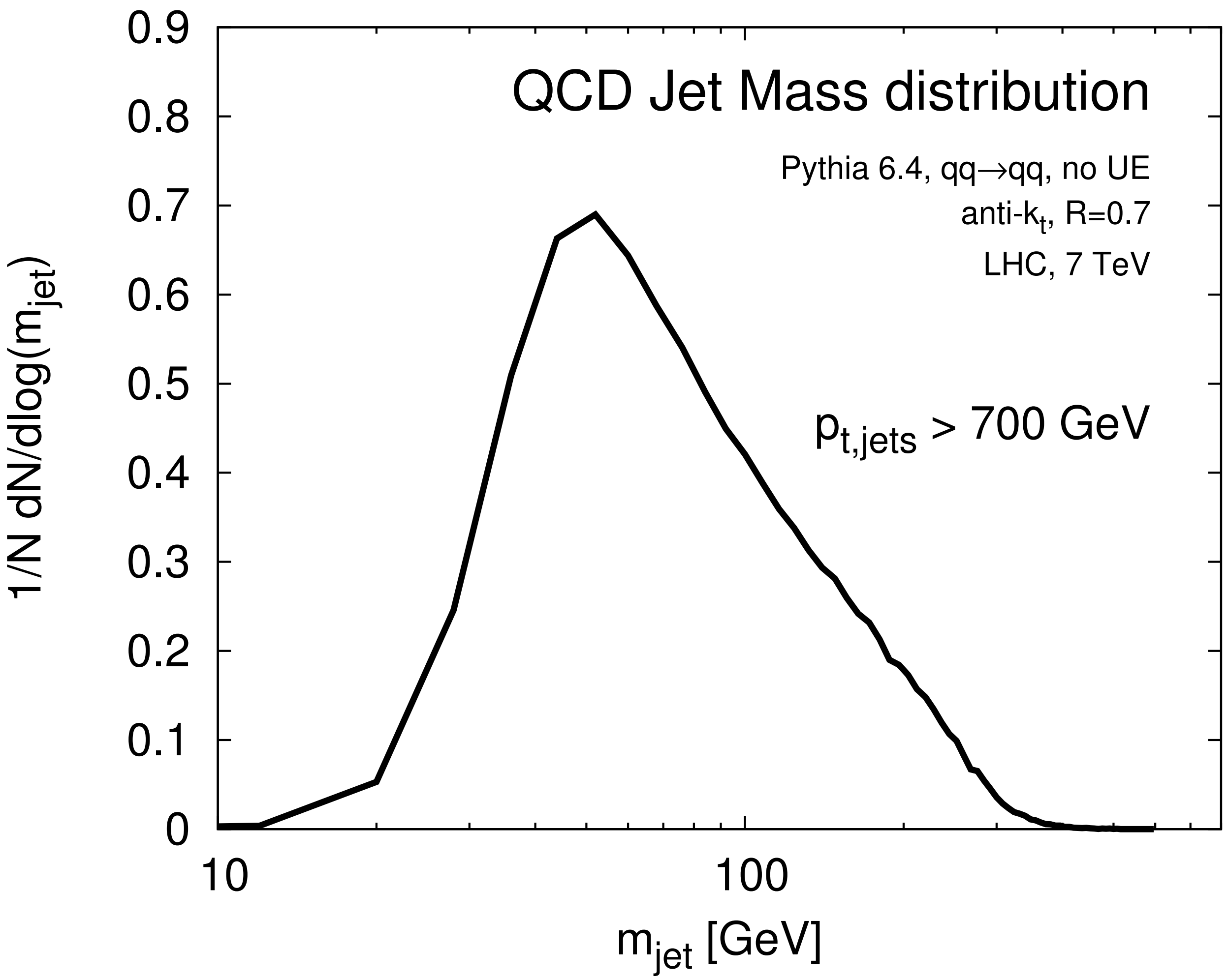
Tagging

- reduces the background, leaves much of signal
- you can tag with underlying hard n-prong structure and based on radiation pattern

Grooming

- improves signal mass resolution (removing pileup, etc.), without significantly changing background & signal event numbers

One core idea for tagging

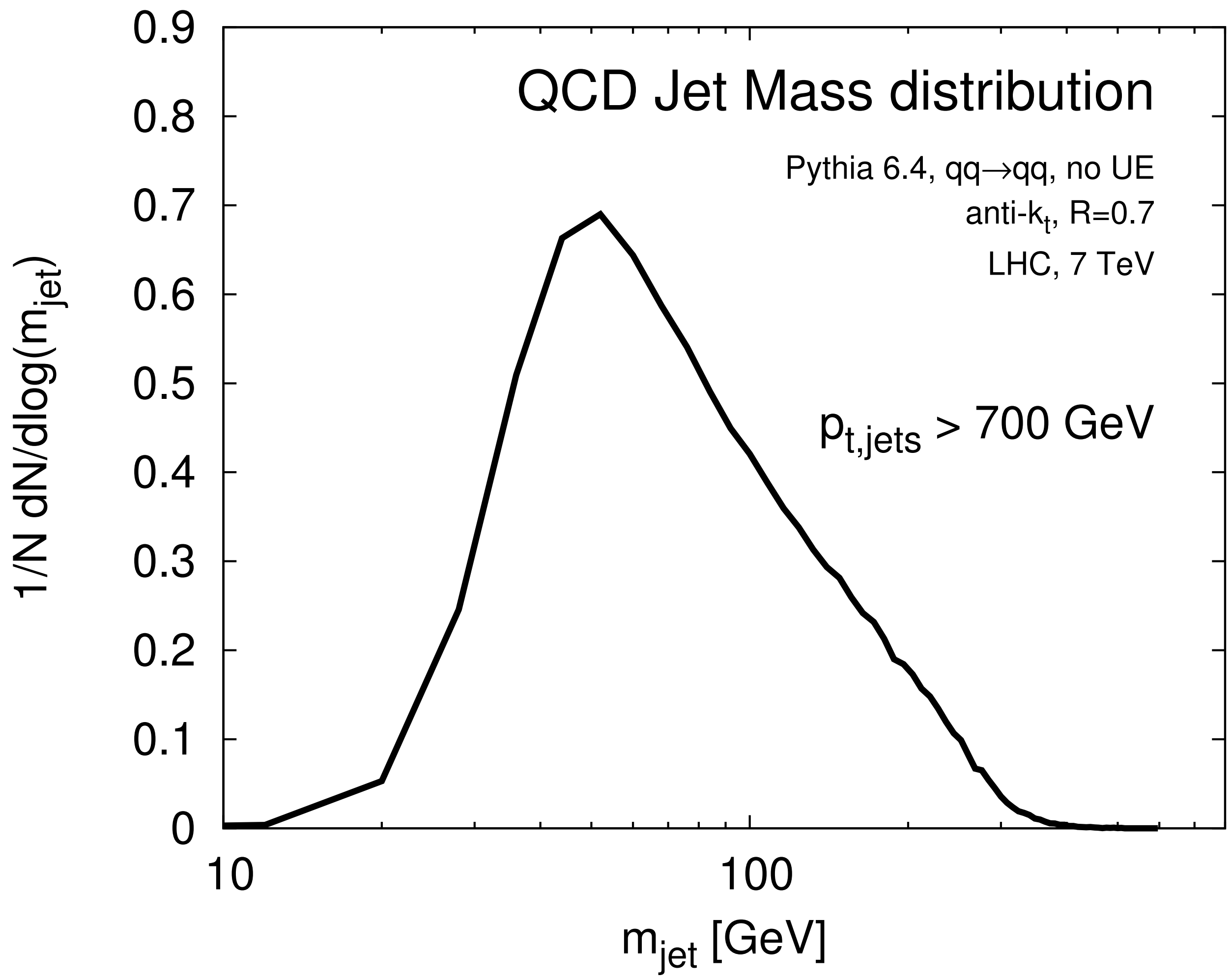


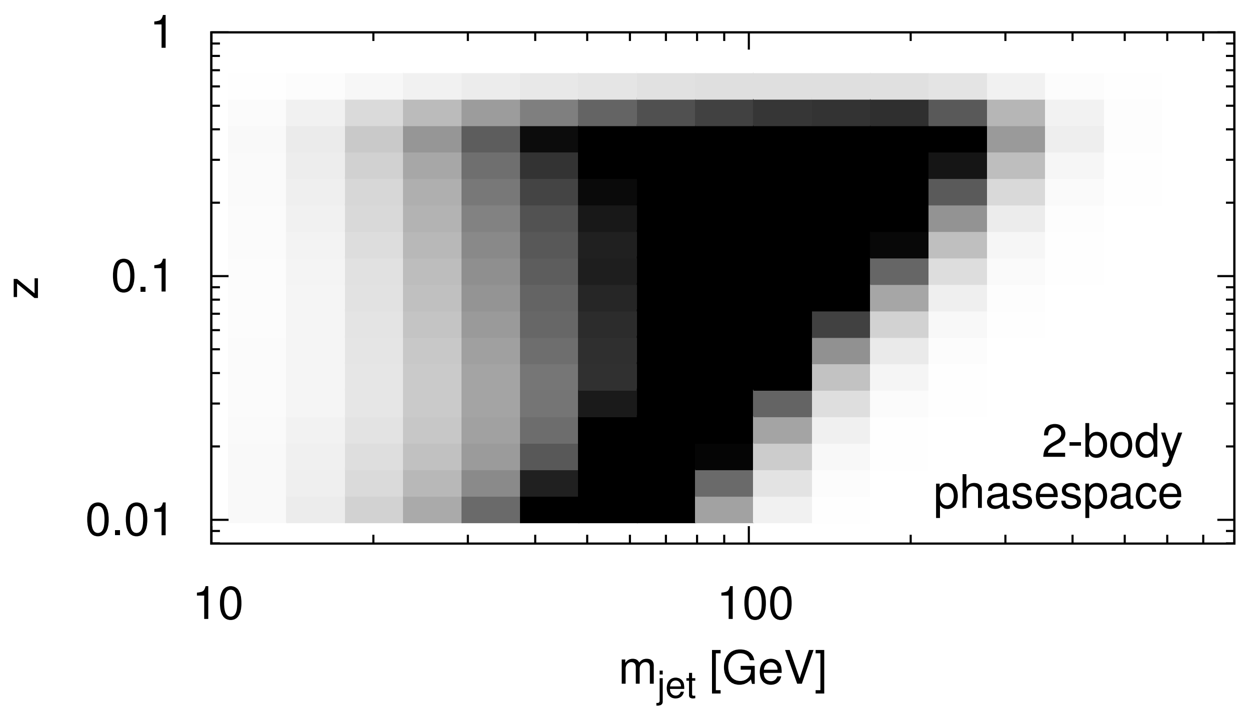
QCD jet mass distribution has the approximate

$$\frac{dN}{d\ln m} \sim \alpha_s \ln\frac{p_t R}{m} \times \text{Sudakov}$$

Work from '80s and '90s

Inside the jet mass





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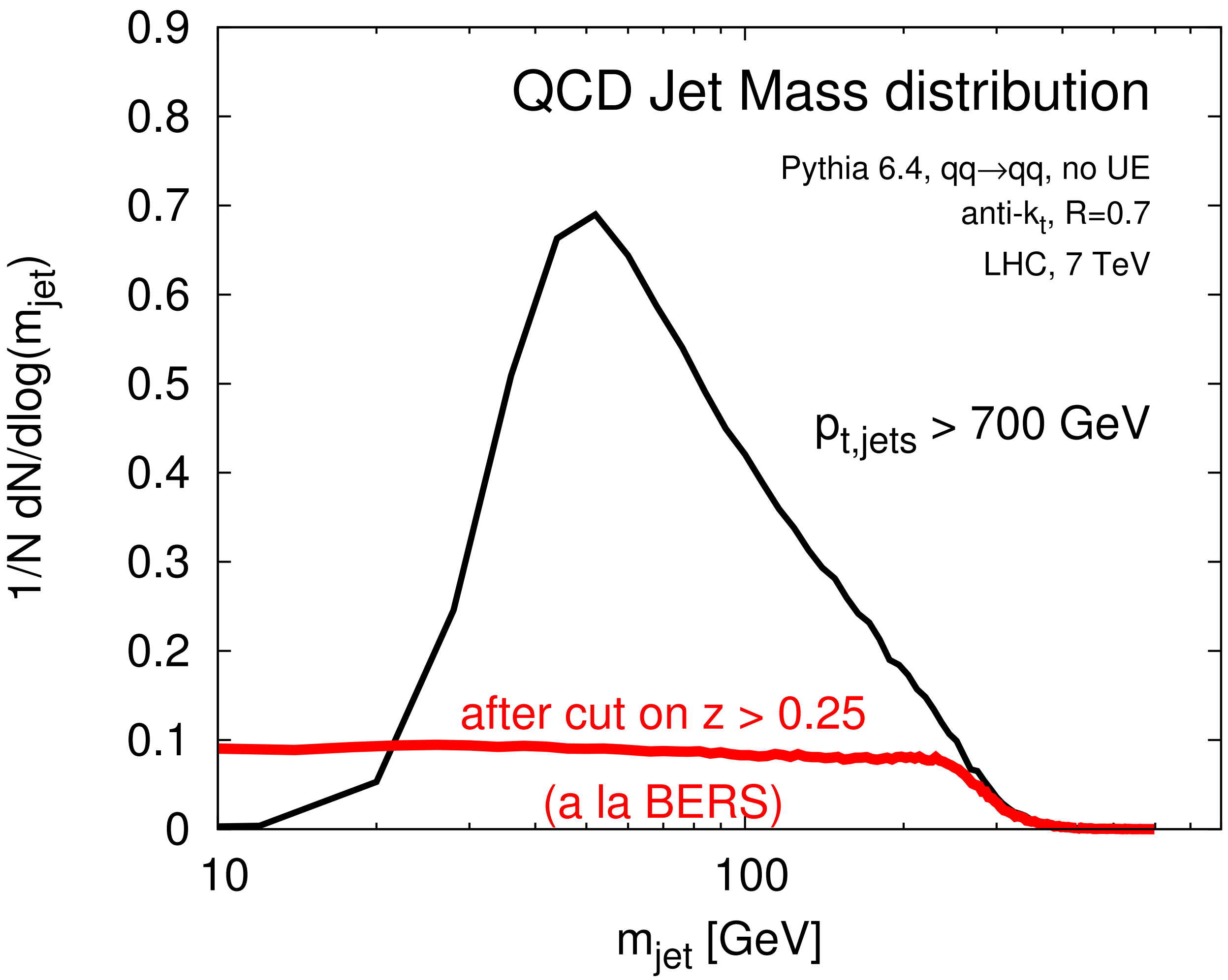
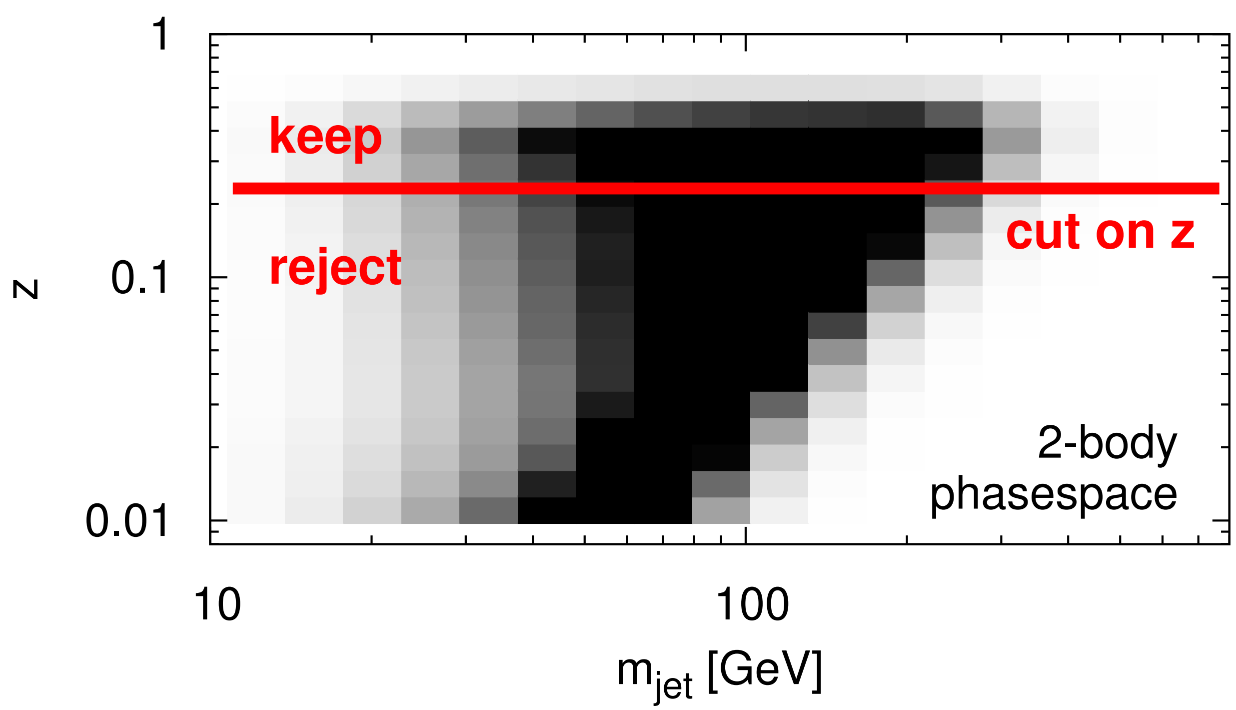
Work from '80s and '90s

The logarithm comes from integral over soft divergence of QCD:

$$\int_{\frac{m^2}{p_t^2 R^2}}^{\frac{1}{2}} \frac{dz}{z}$$





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Work from '80s and '90s

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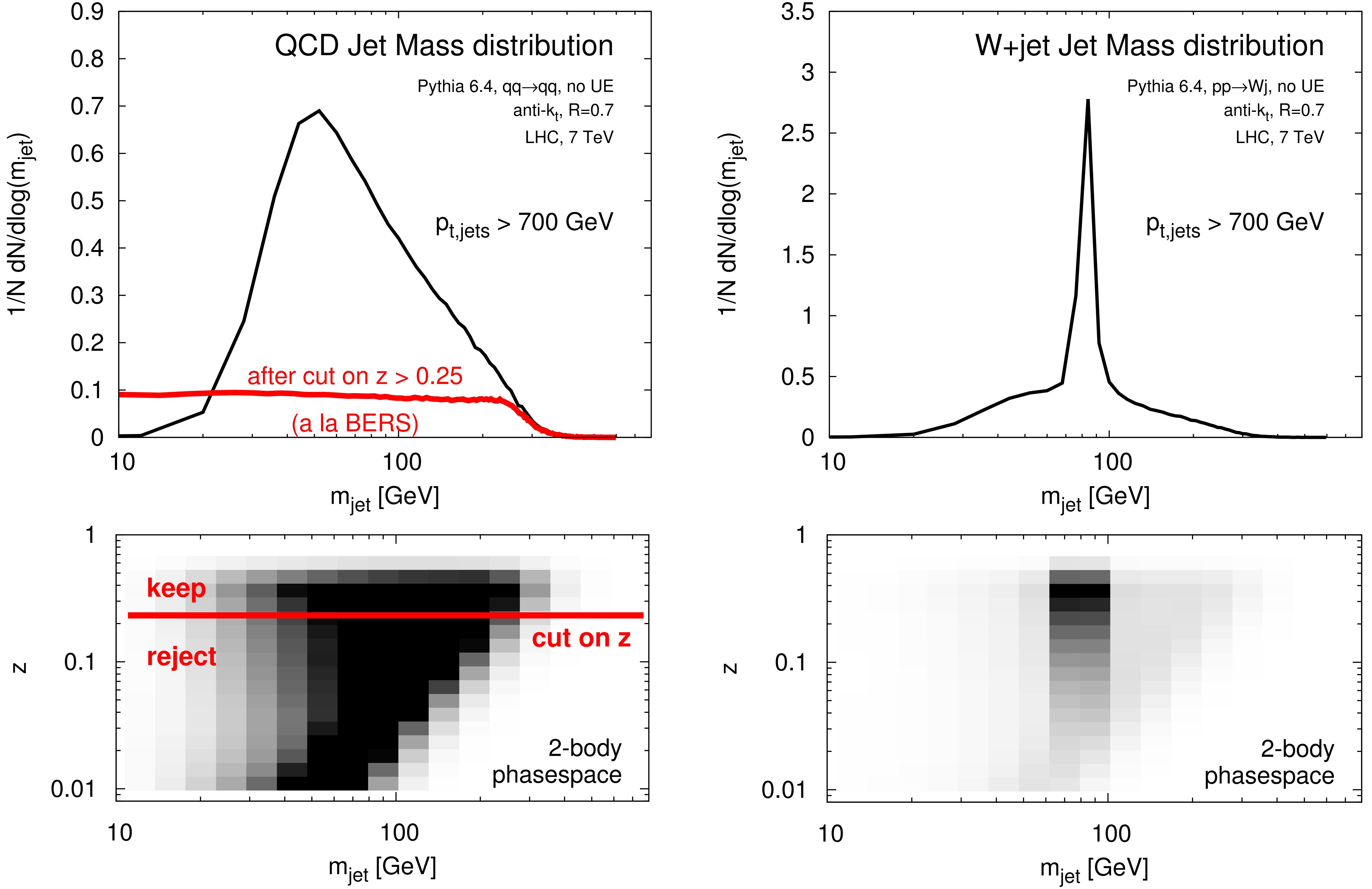
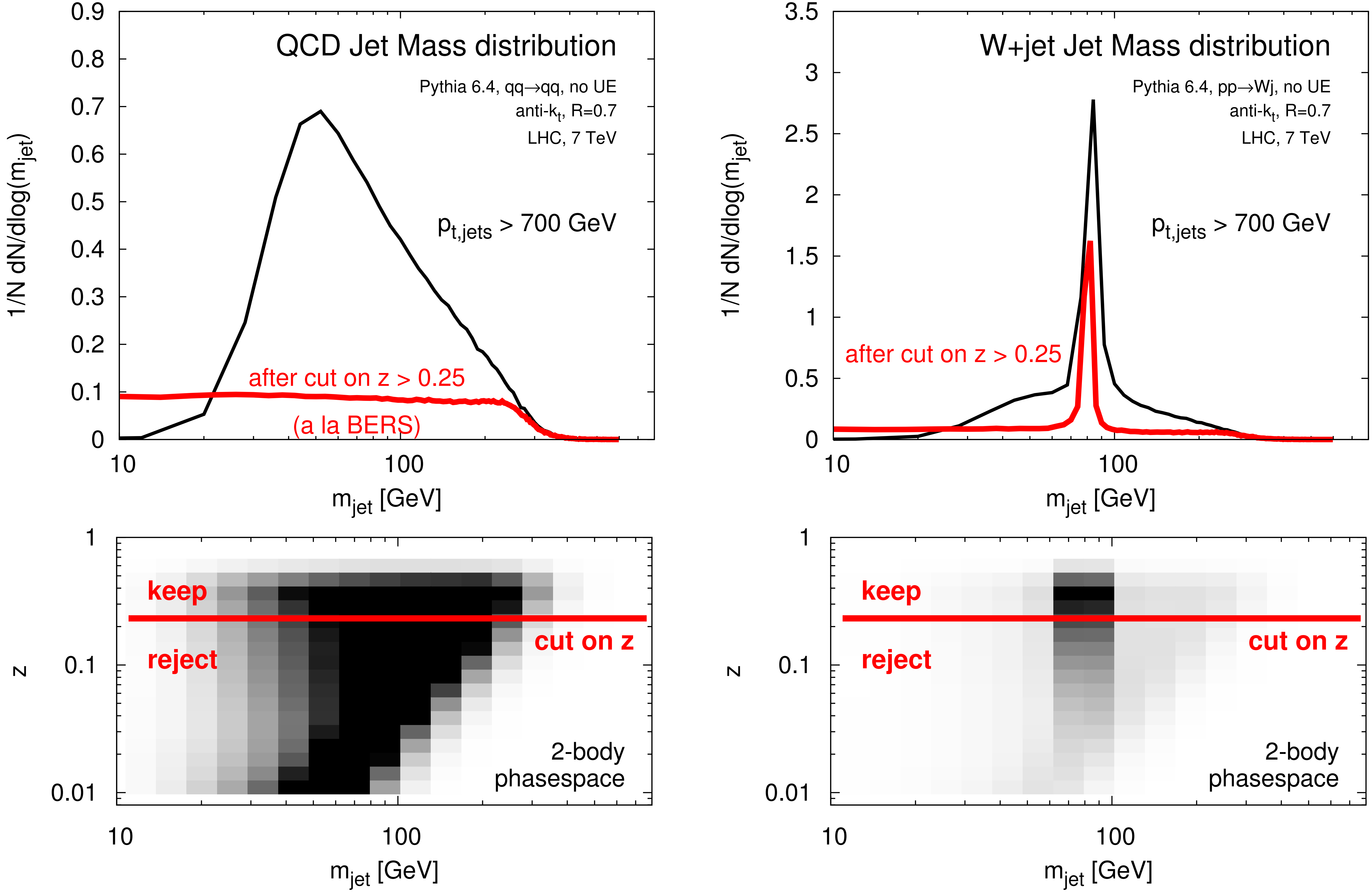
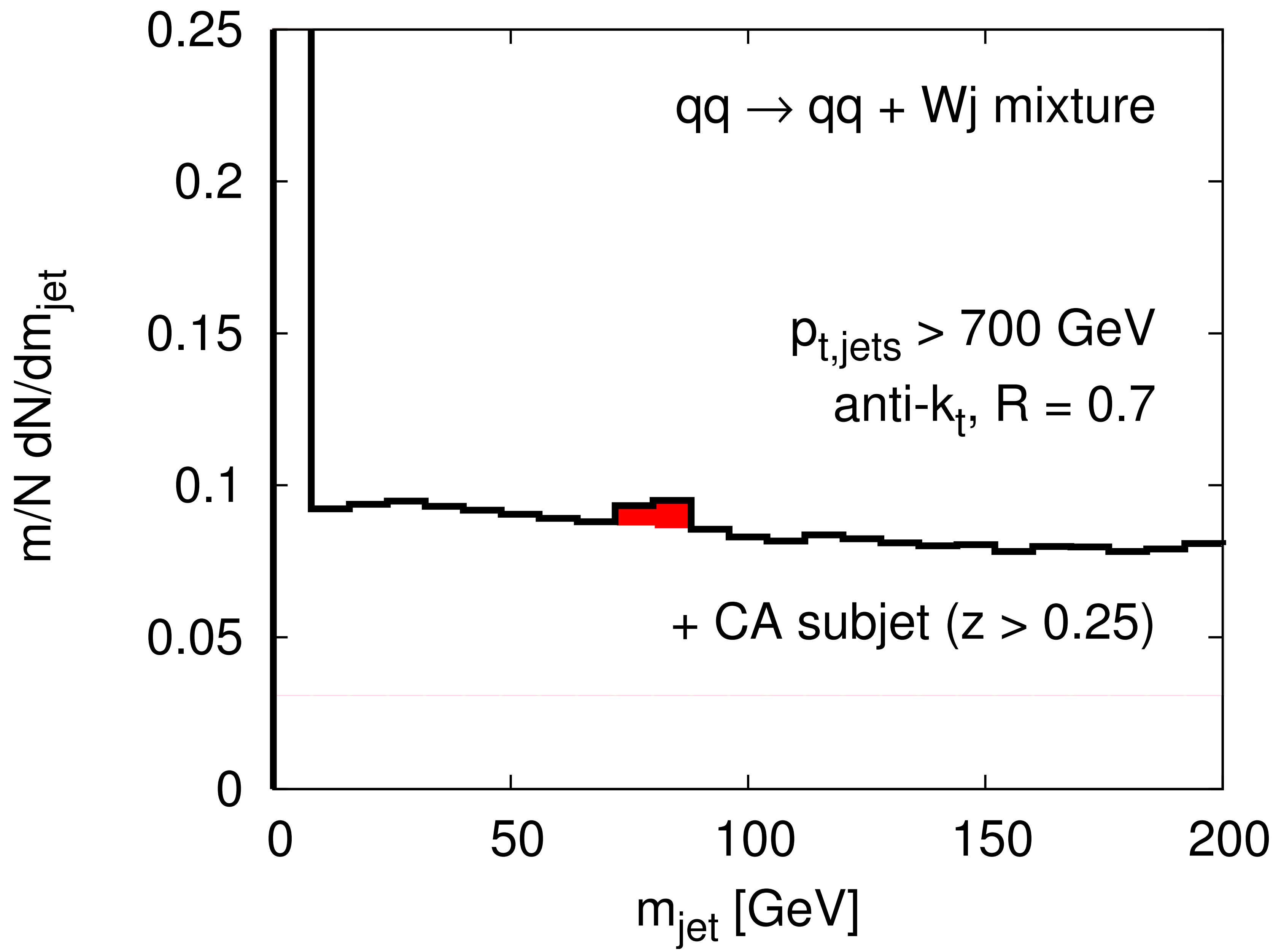
$$\int_{\frac{m^2}{p_t^2 R^2}}^{\frac{1}{2}} \frac{dz}{z}$$

A hard cut on z reduces QCD background & simplifies its shape

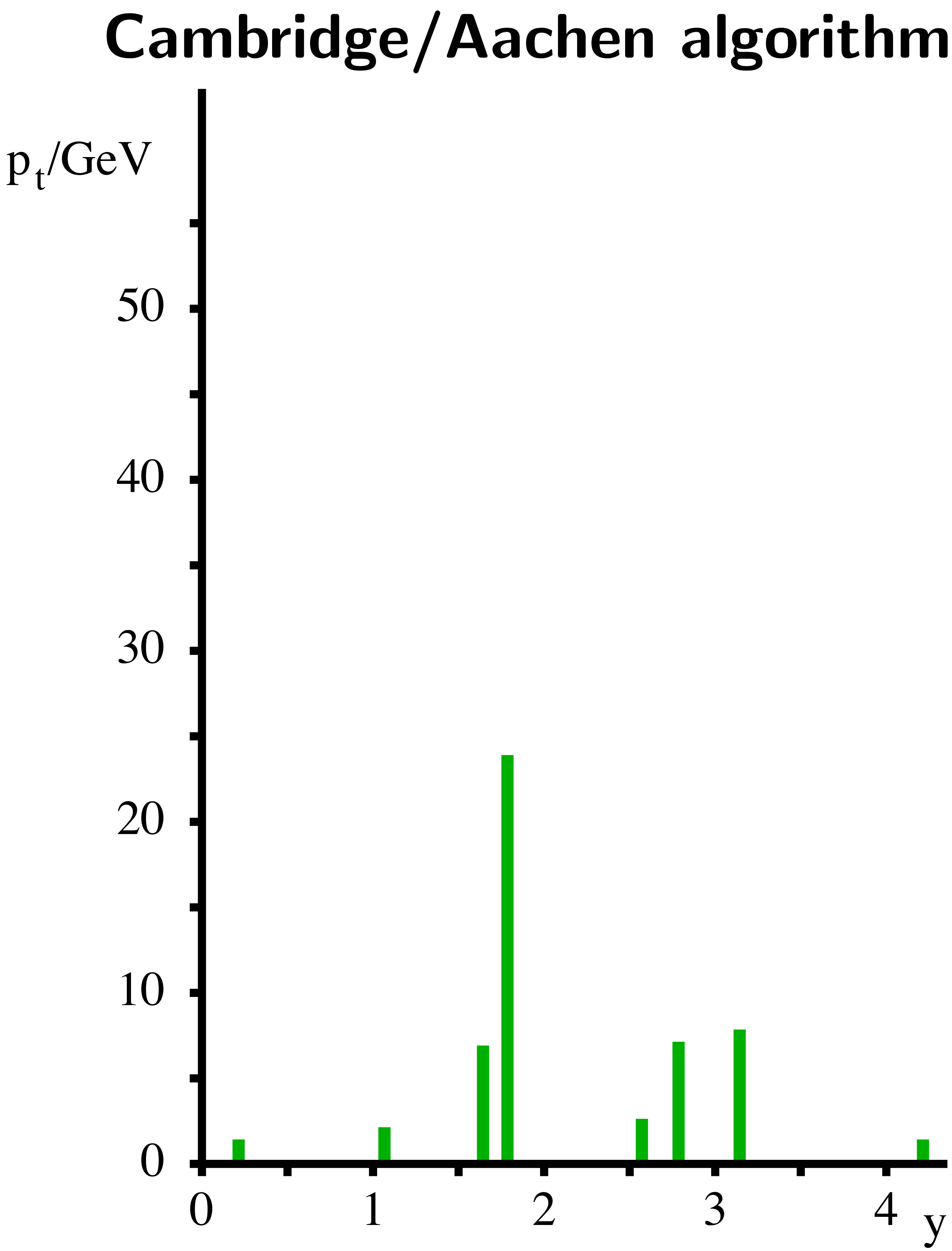

QCD Jet Mass distribution
Pythia 6.4, qq→qq, no UE
anti-kt, R=0.7
LHC, 7 TeV
pt,jets > 700 GeV
after cut on z > 0.25
(a la BERS)
1/N dN/dlog(mjet)
mjet [GeV]
W+jet Jet Mass distribution
Pythia 6.4, pp→Wj, no UE
anti-kt, R=0.7
LHC, 7 TeV
pt,jets > 700 GeV
1/N dN/dlog(mjet)
mjet [GeV]
keep
reject
cut on z
2-body
phasespace
z
mjet [GeV]
2-body
phasespace
z
mjet [GeV]

Inside the jet mass

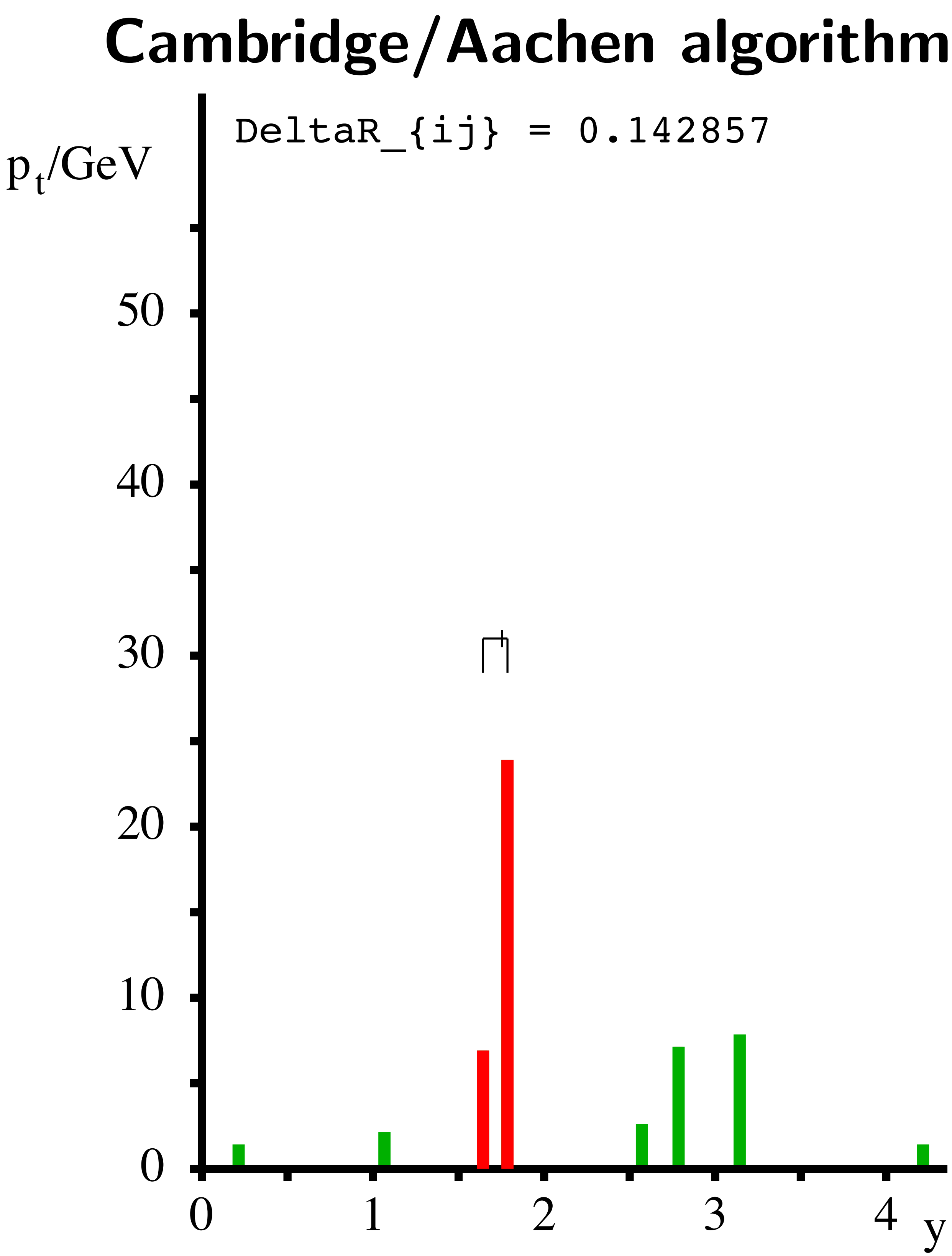




Signal + bkgd
after cut on z



How well can an algorithm identify the "blobs" of energy inside a jet that come from different partons?



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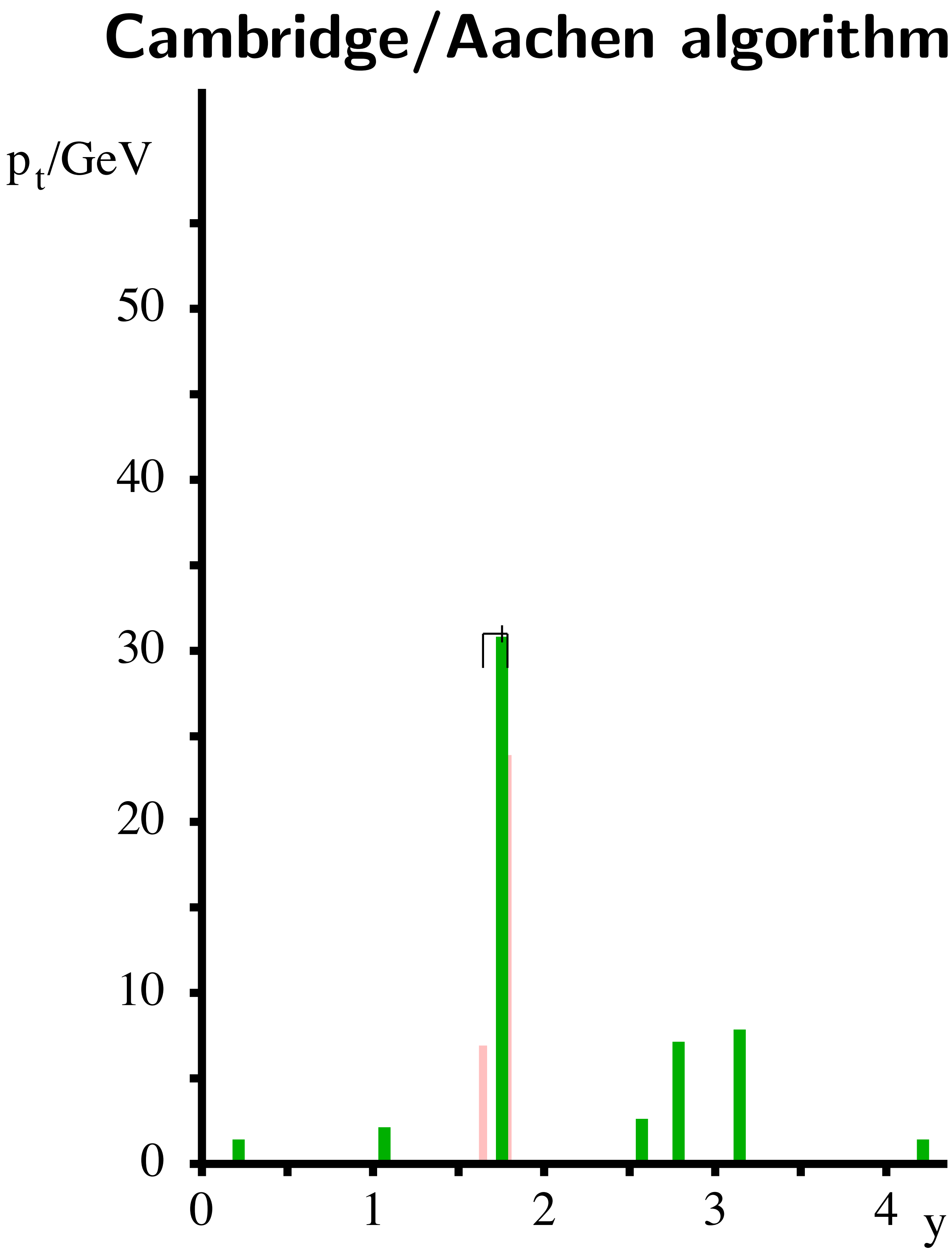

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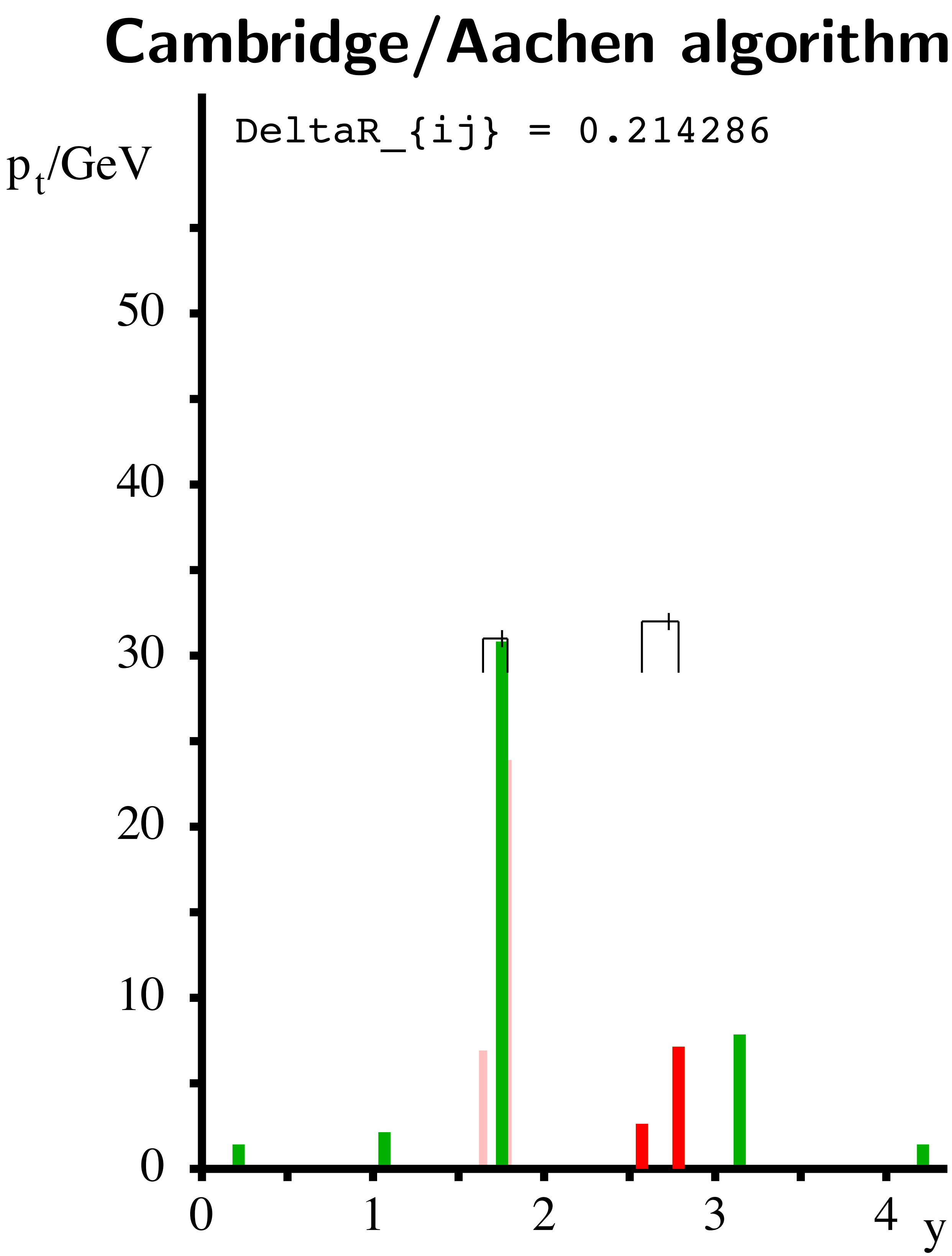

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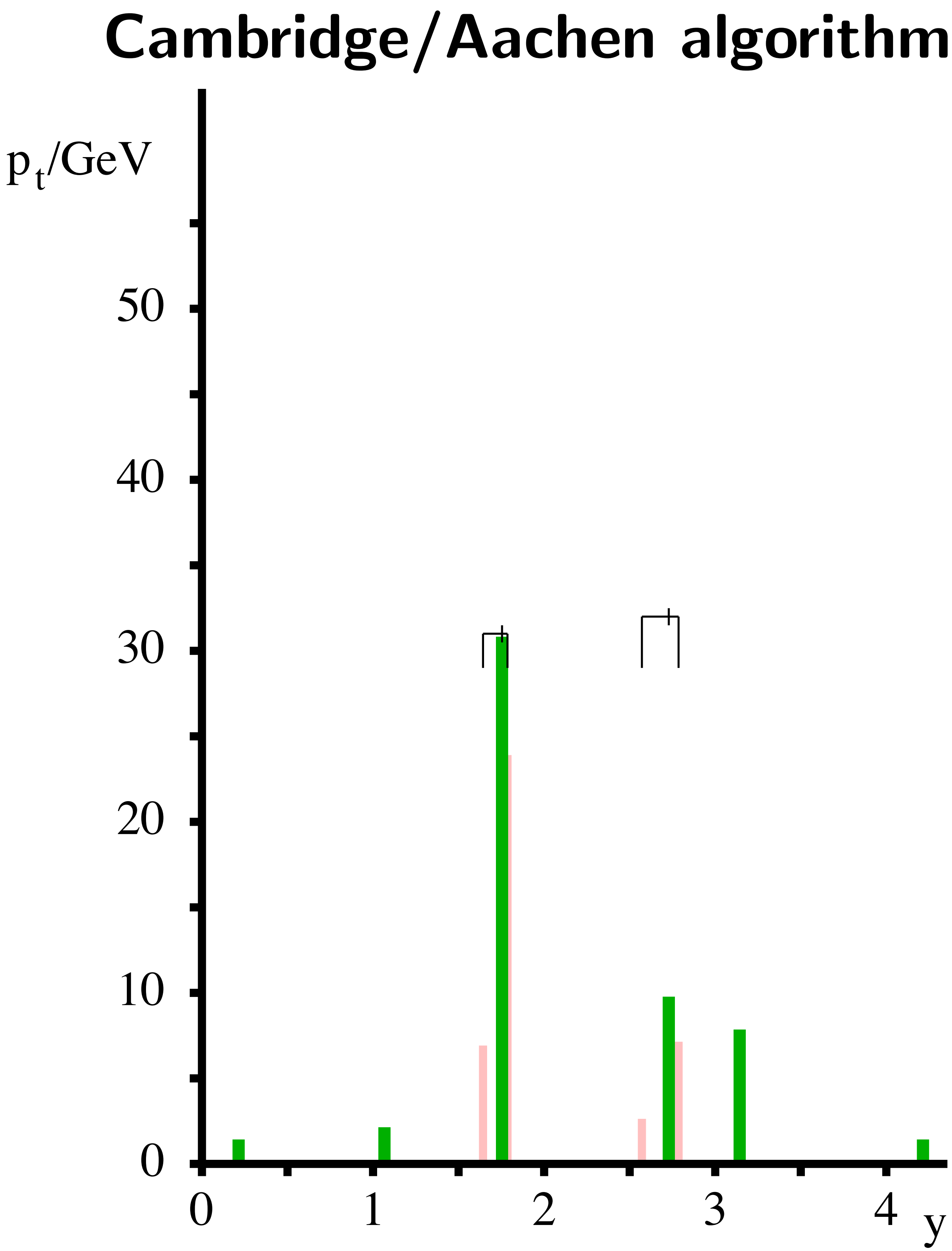

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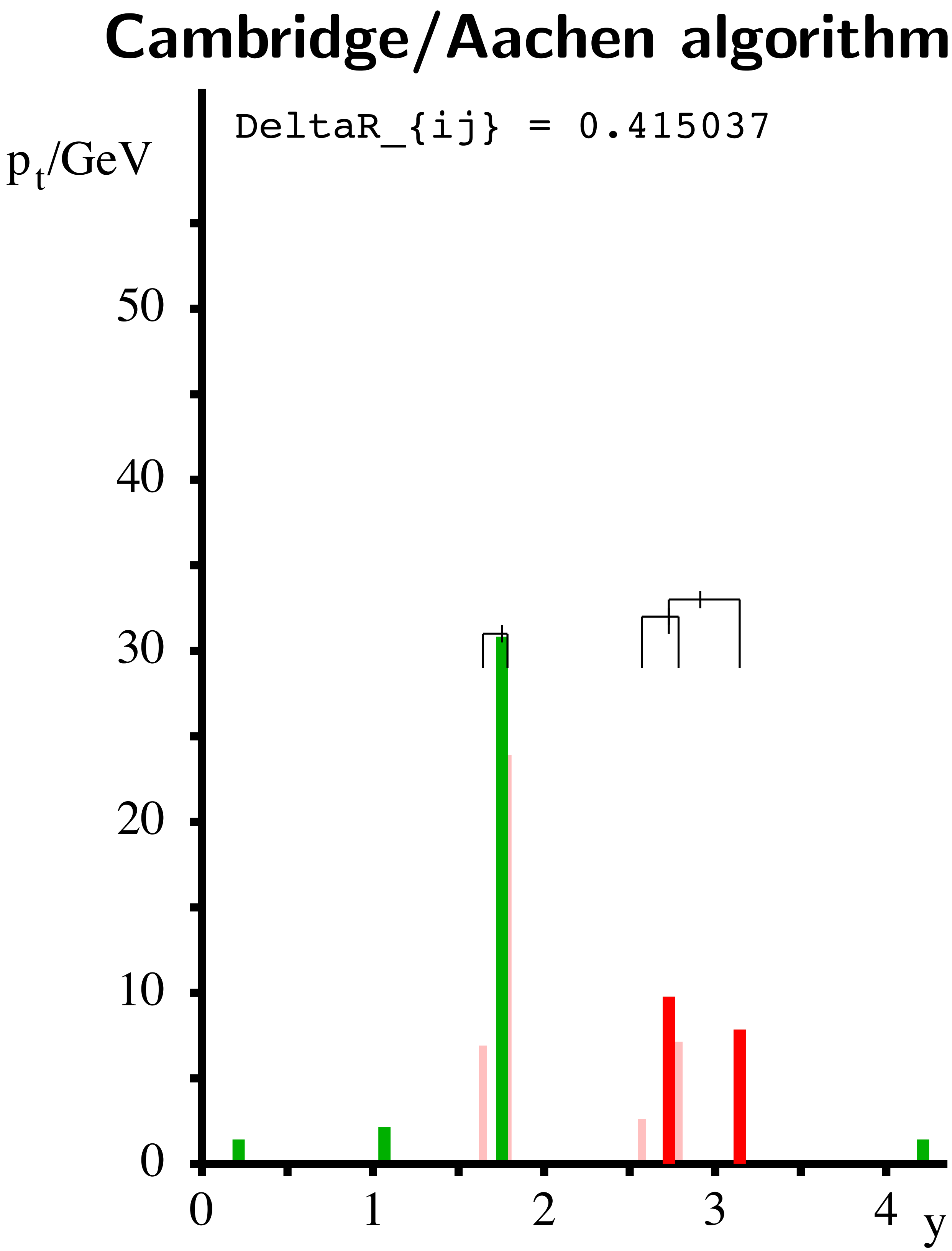

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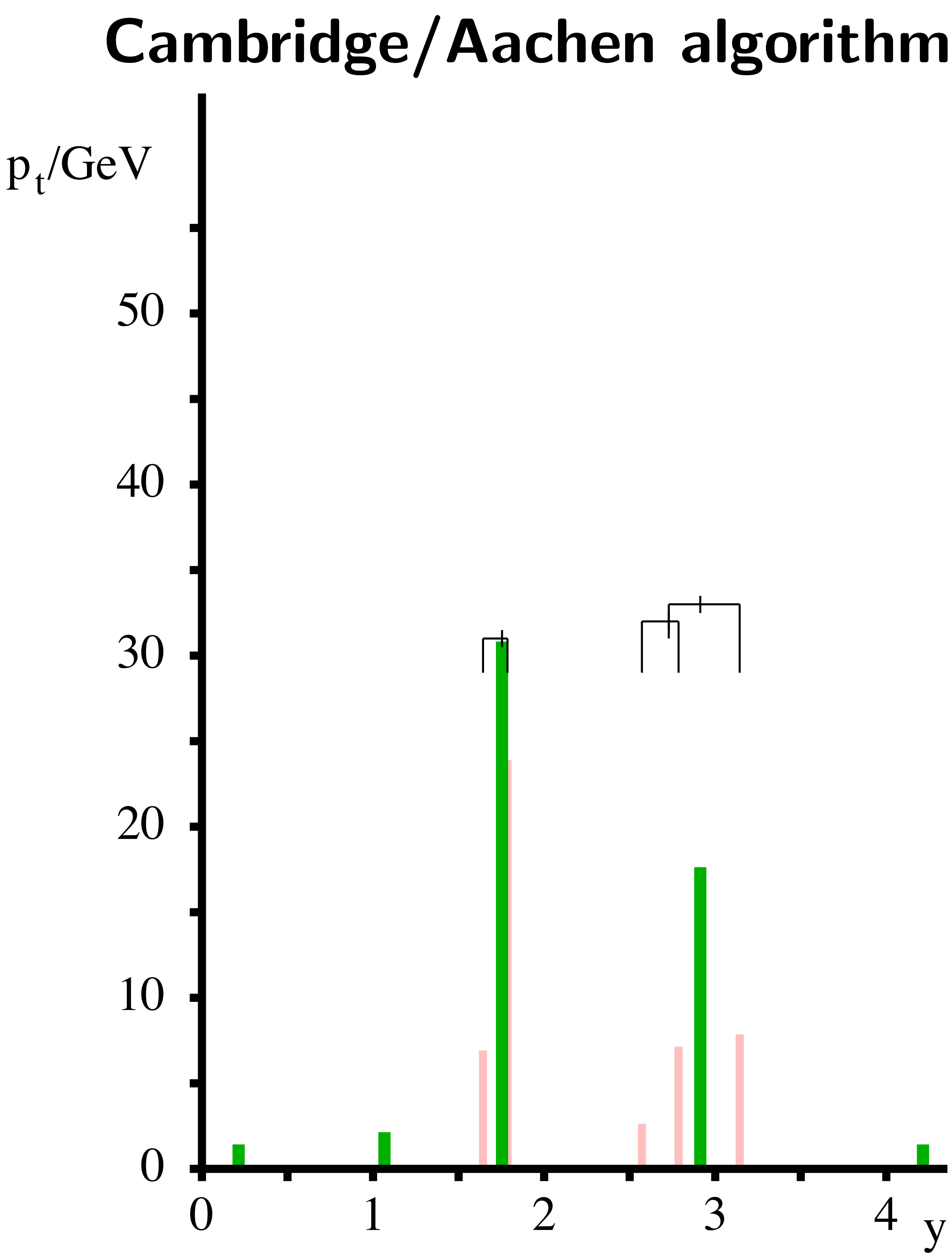

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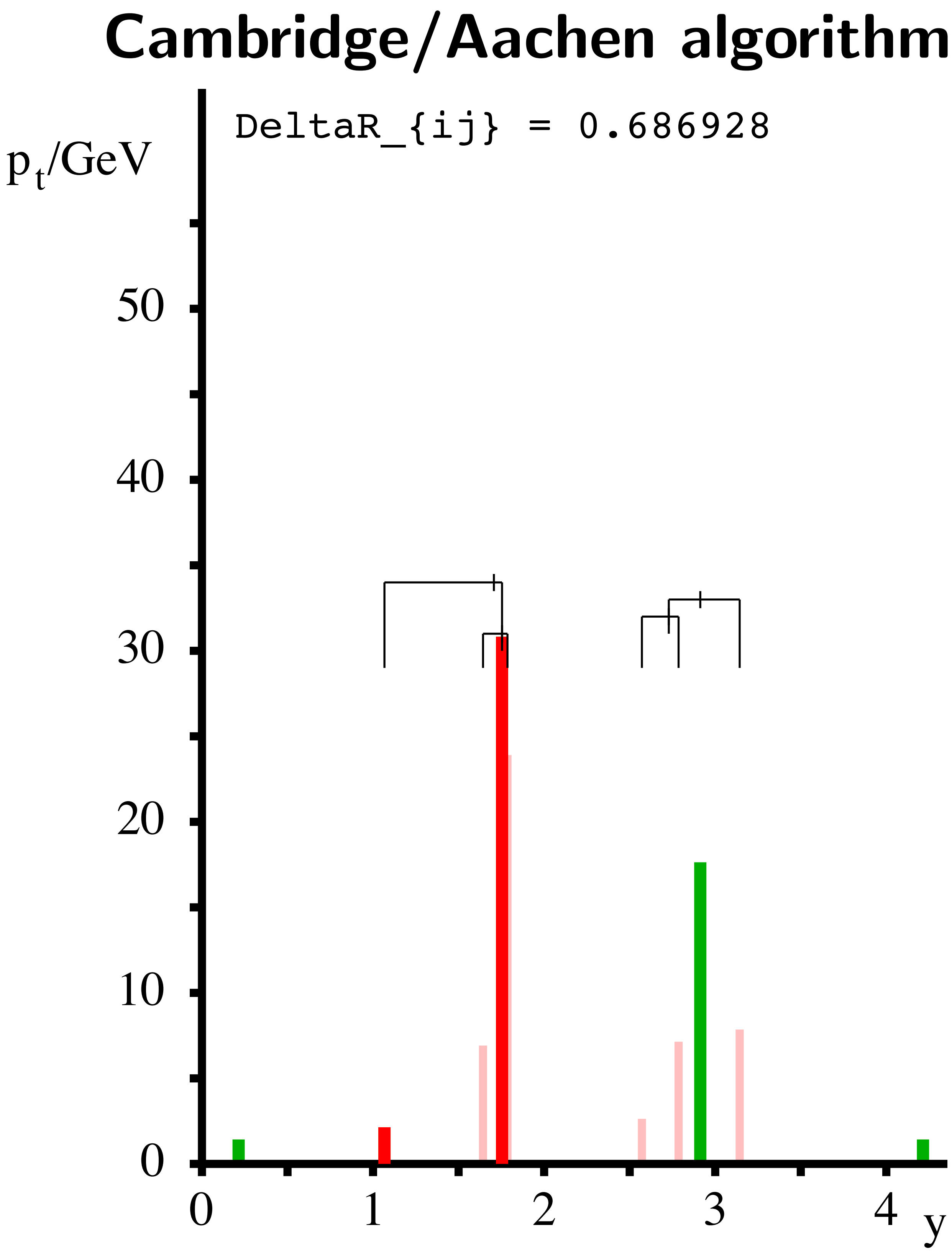

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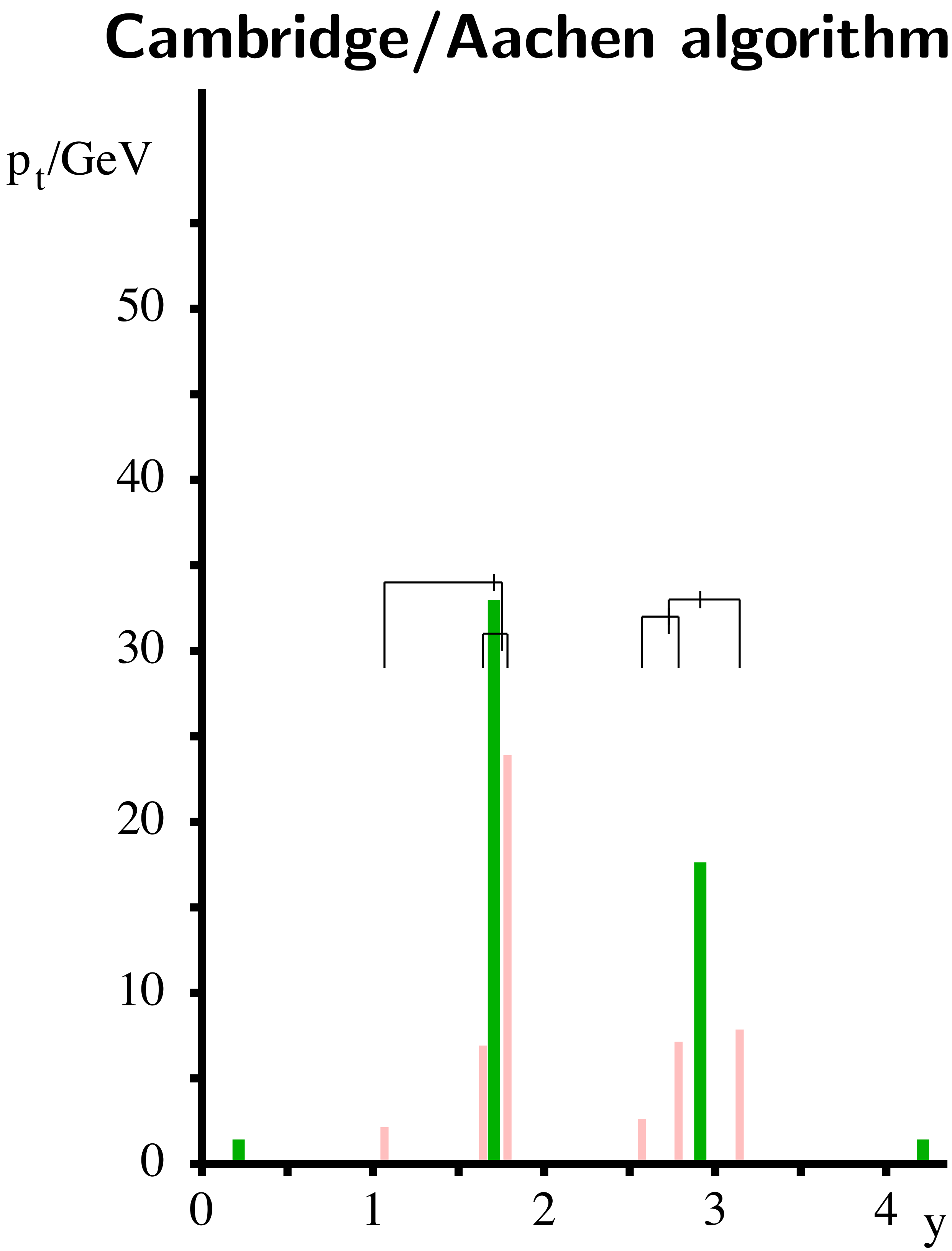
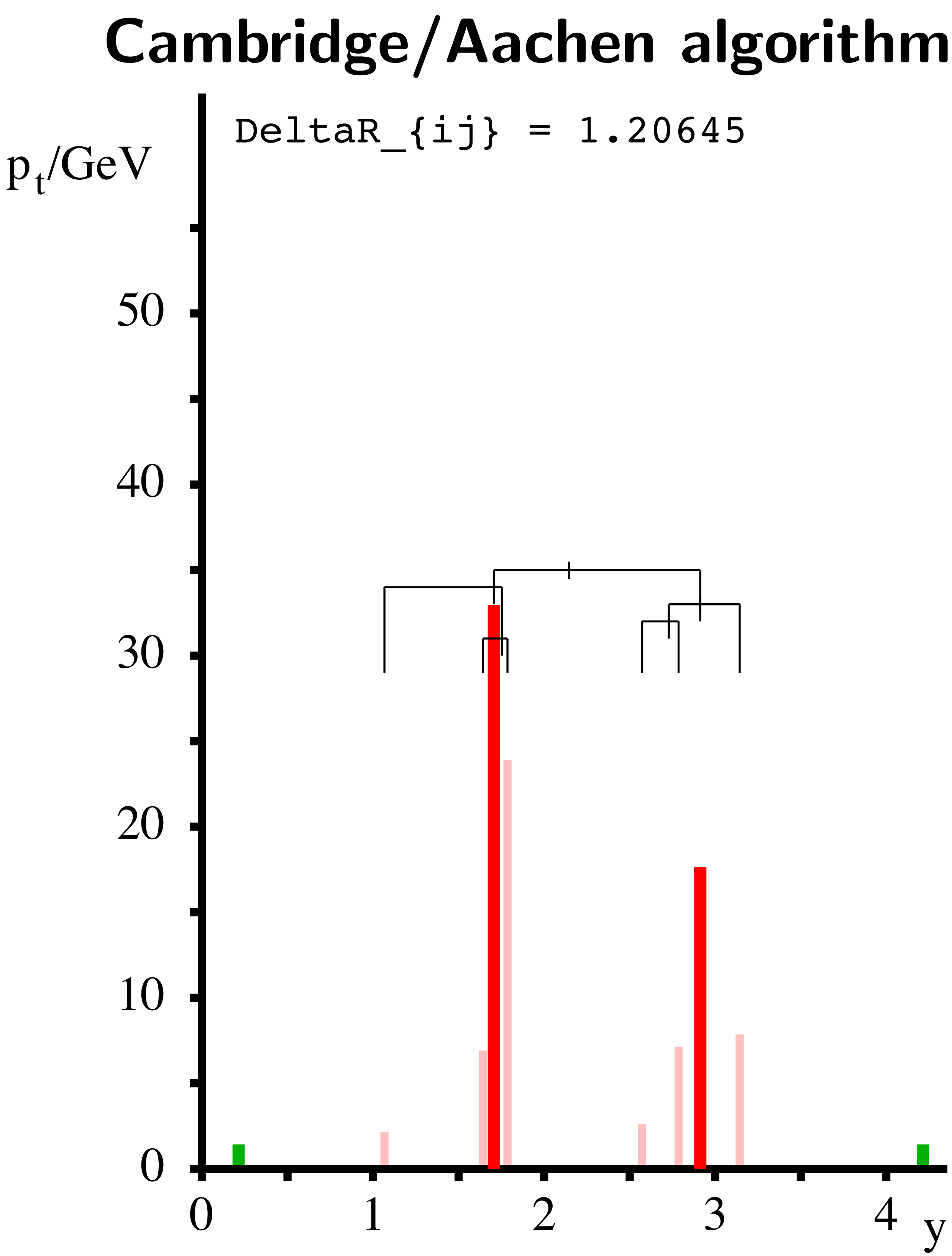


How well can an algorithm identify the “blobs” of energy inside a jet that come from different partons?

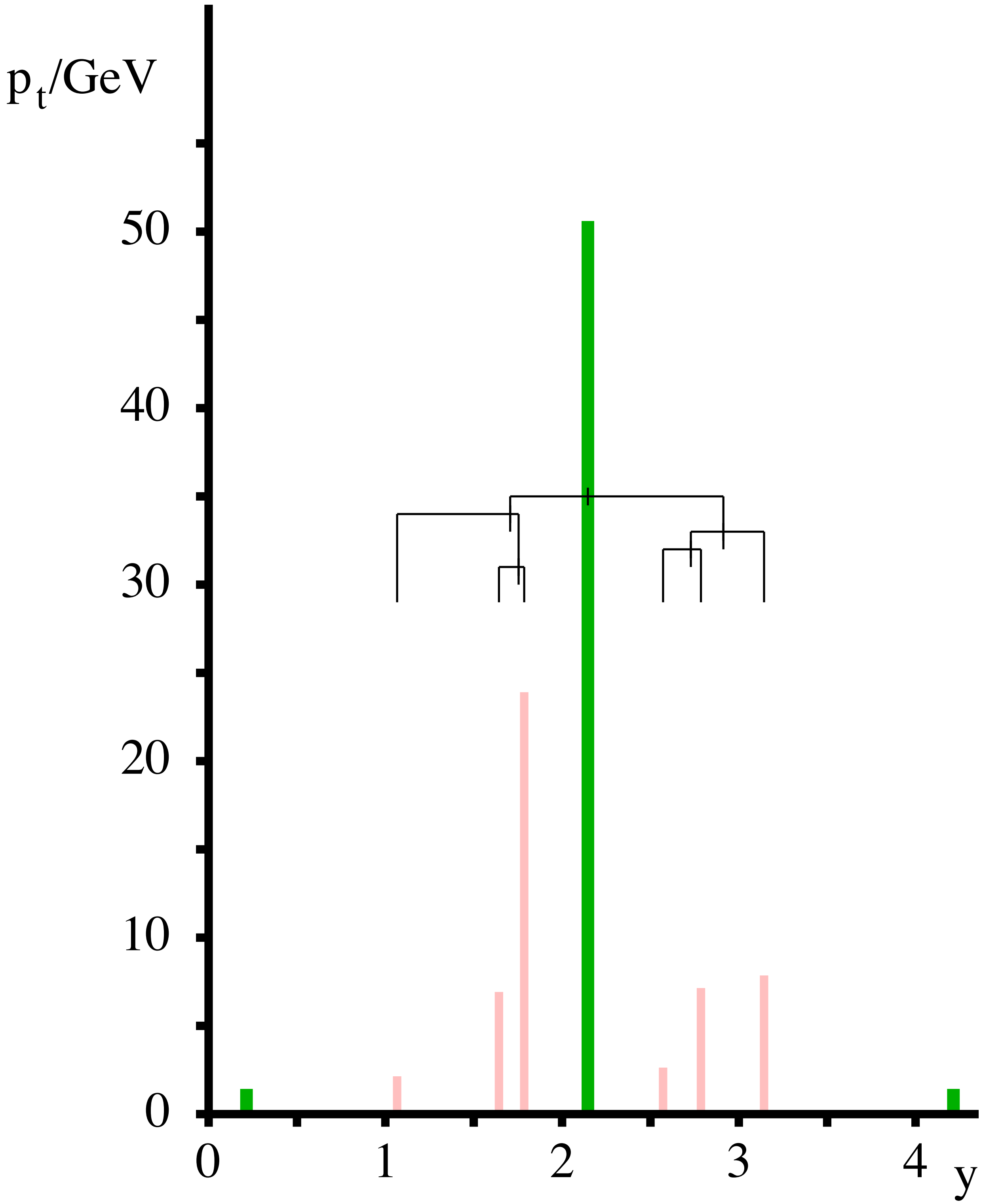
C/A identifies two hard blobs with limited soft contamination



How well can an algorithm identify the "blobs" of energy inside a jet that come from different partons?

C/A identifies two hard blobs with limited soft contamination, joins them

Cambridge/Aachen algorithm



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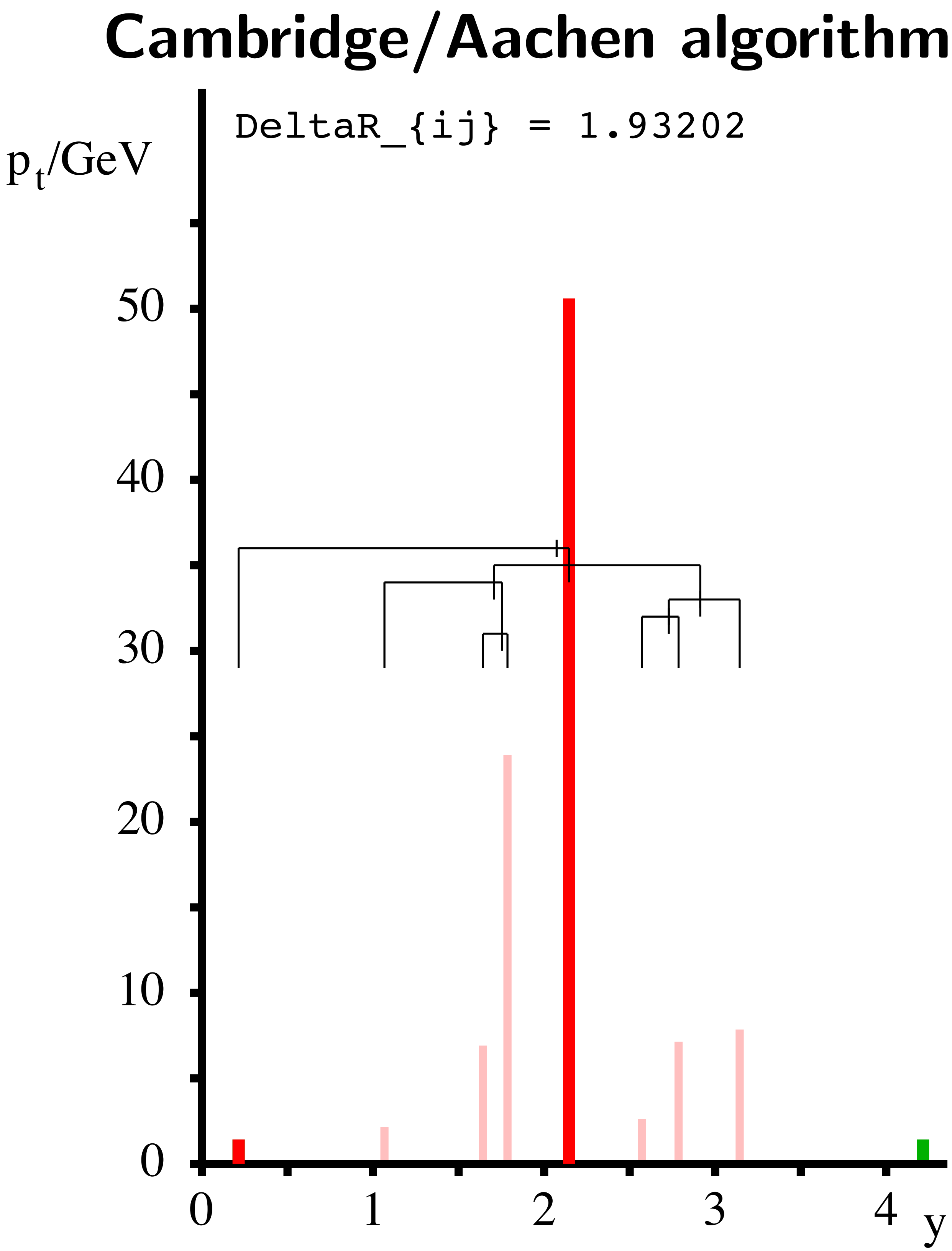
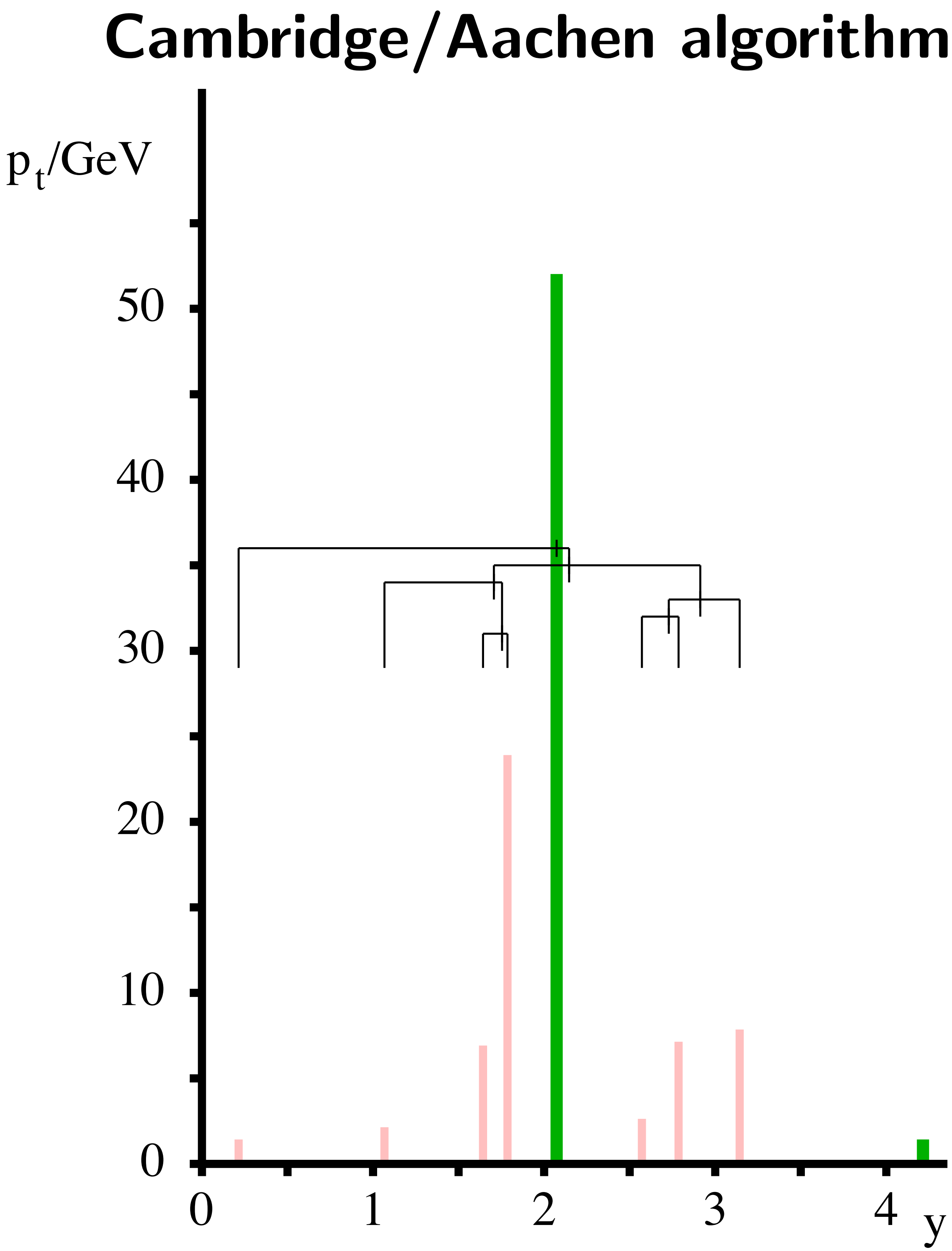


How well can an algorithm identify the "blobs" of energy inside a jet that come from different partons?

C/A identifies two hard blobs with limited soft contamination, joins them, and then adds in remaining soft junk



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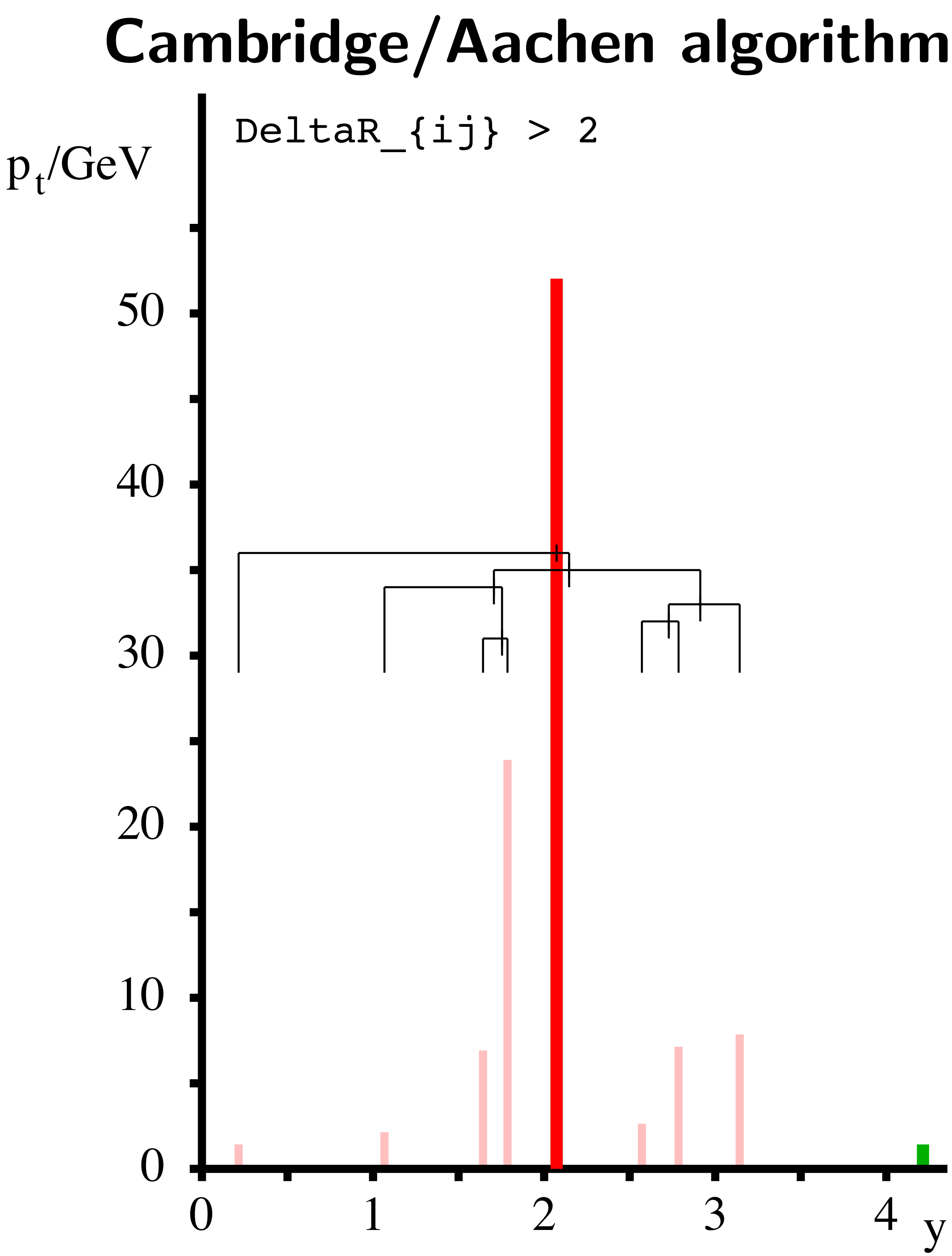

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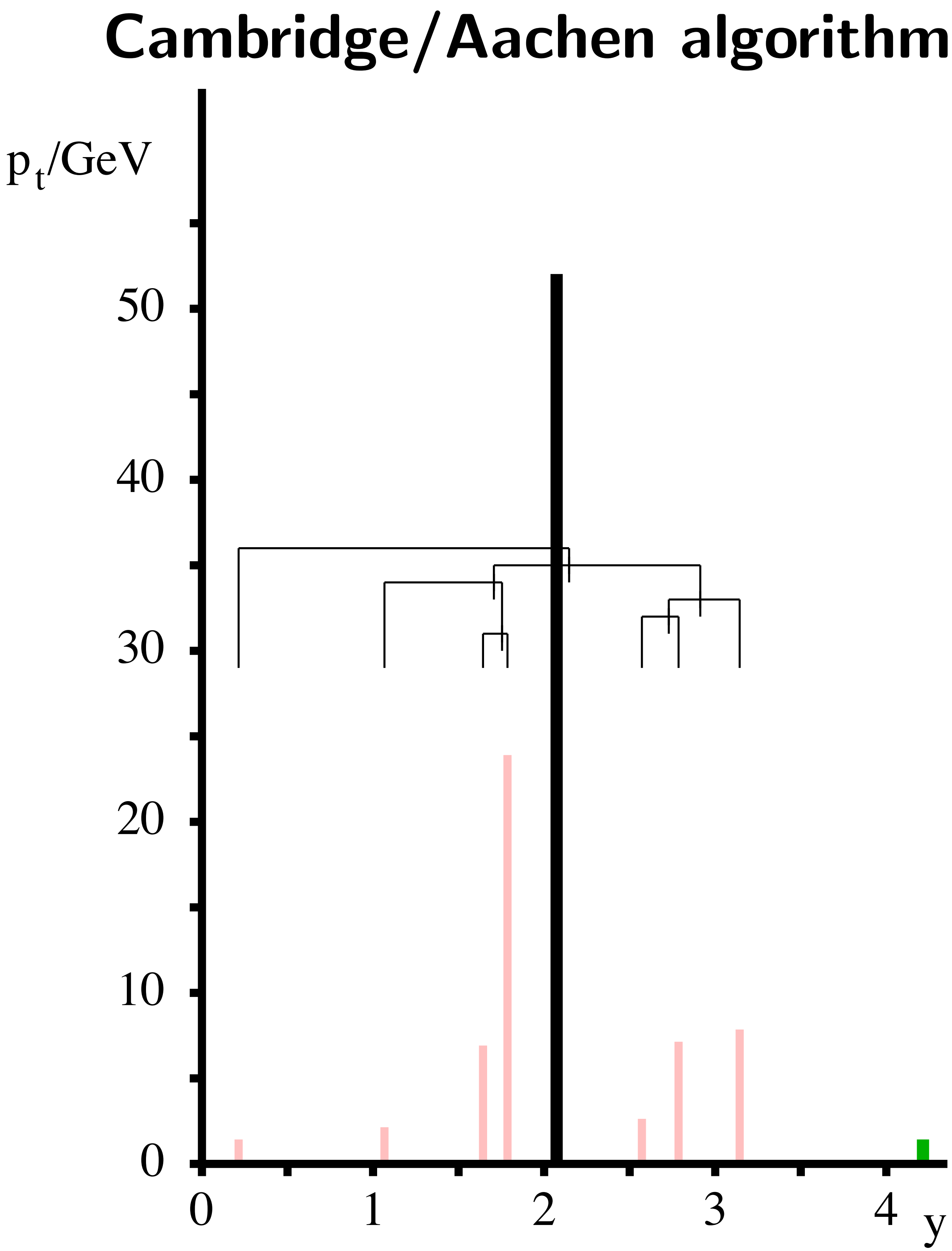

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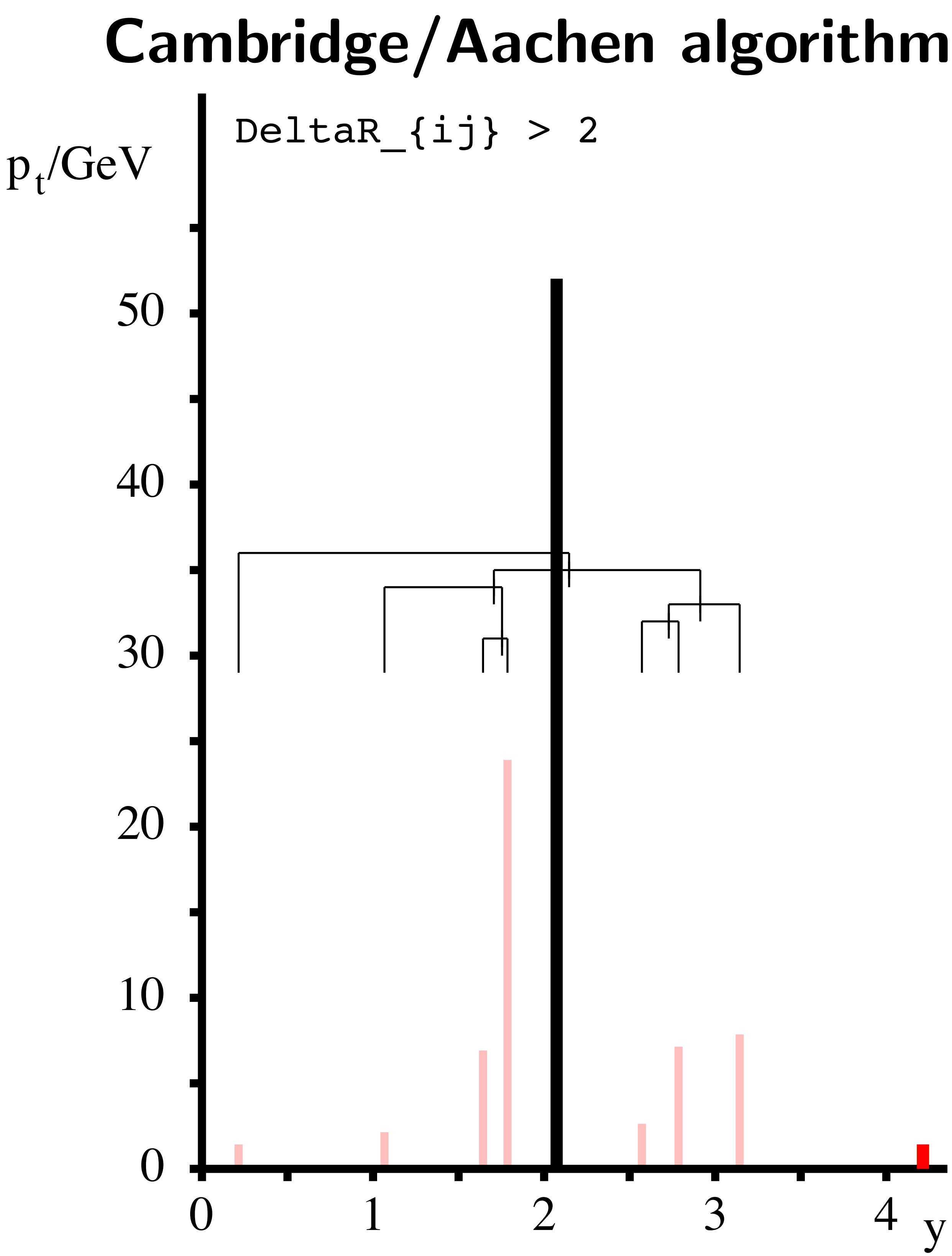

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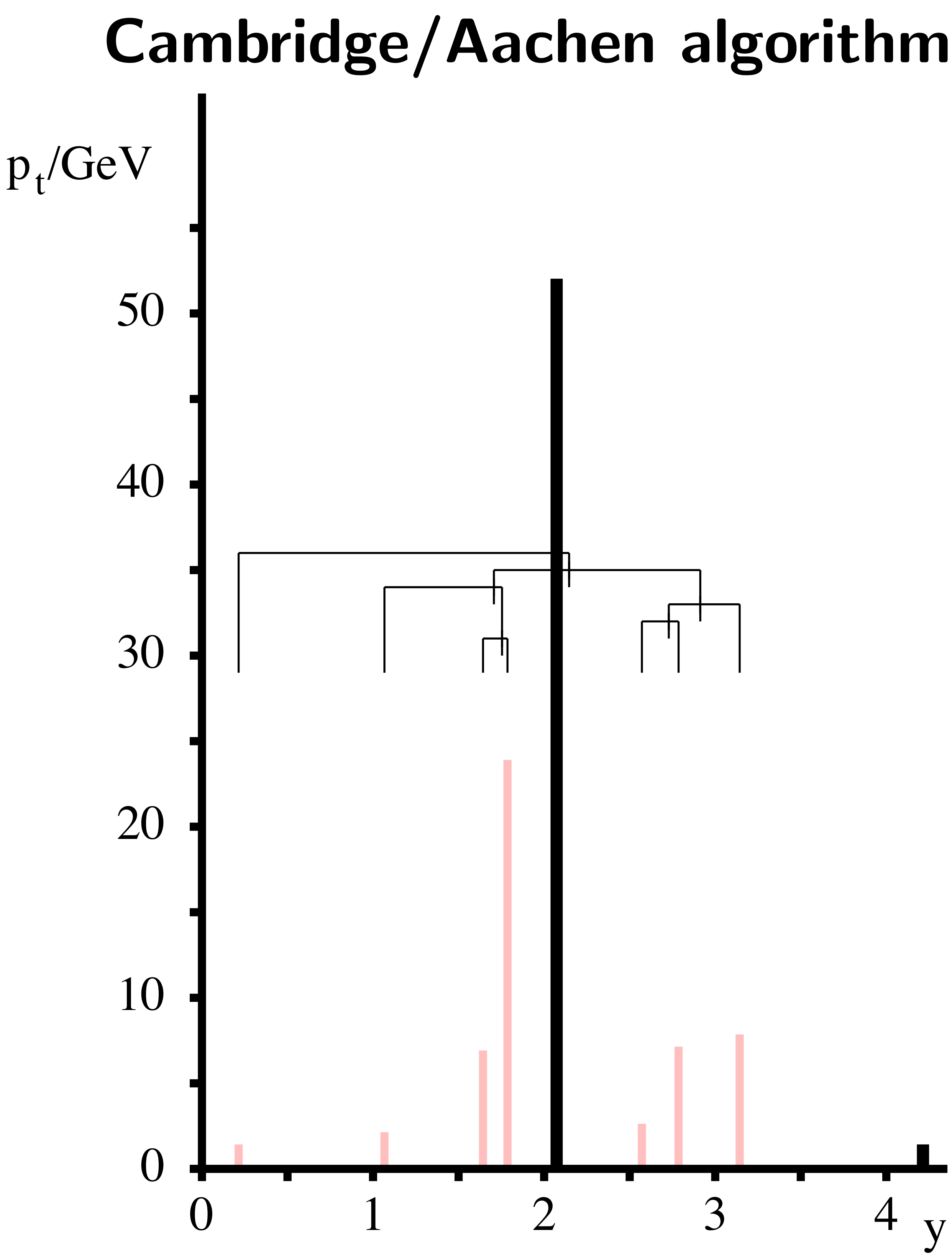

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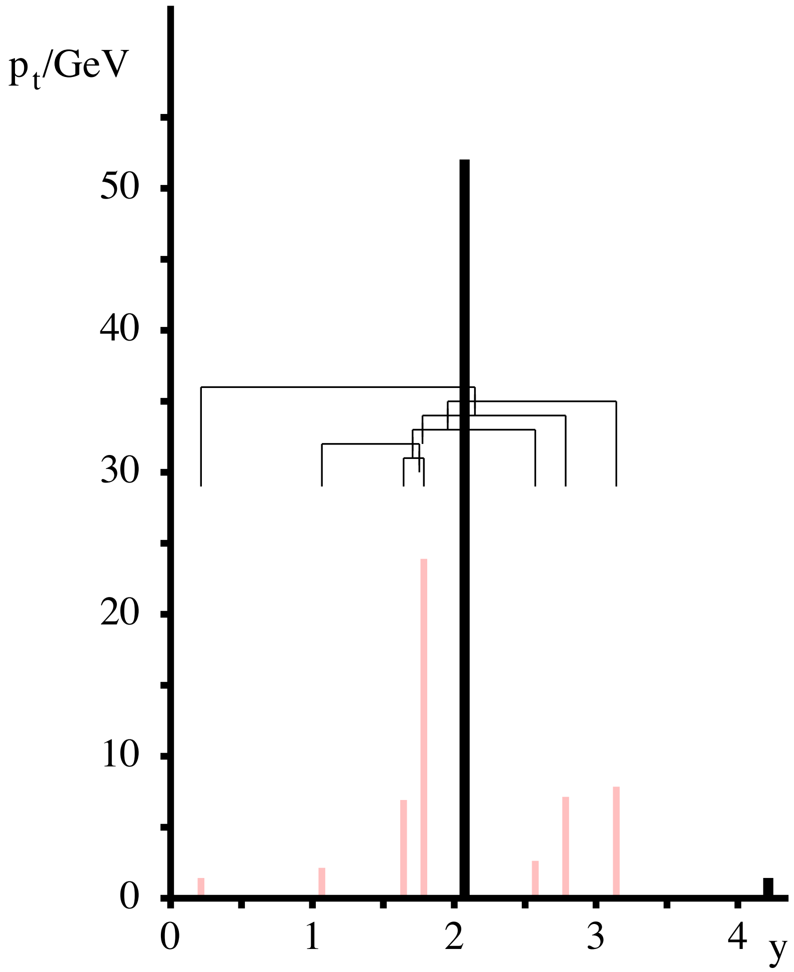
C/A identifies two hard blobs with limited soft contamination, joins them, and then adds in remaining soft junk

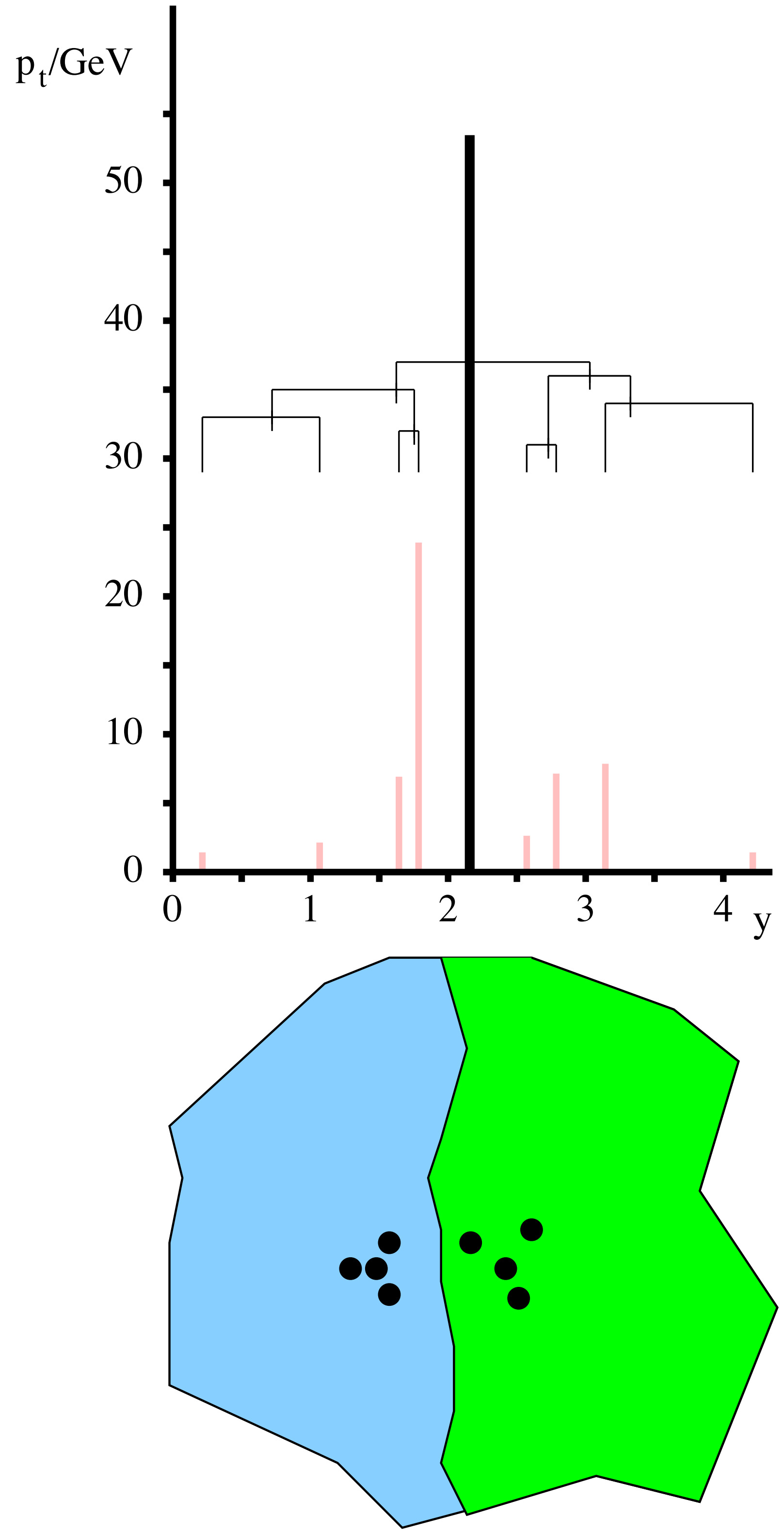
The interesting substructure is buried inside the clustering sequence — it's less contaminated by soft junk, but needs to be pulled out with special techniques

Butterworth, Davison, Rubin & GPS '08
Kaplan, Schwartz, Reherman & Tweedie '08
Butterworth, Ellis, Rubin & GPS '09
Ellis, Vermilion & Walsh '09

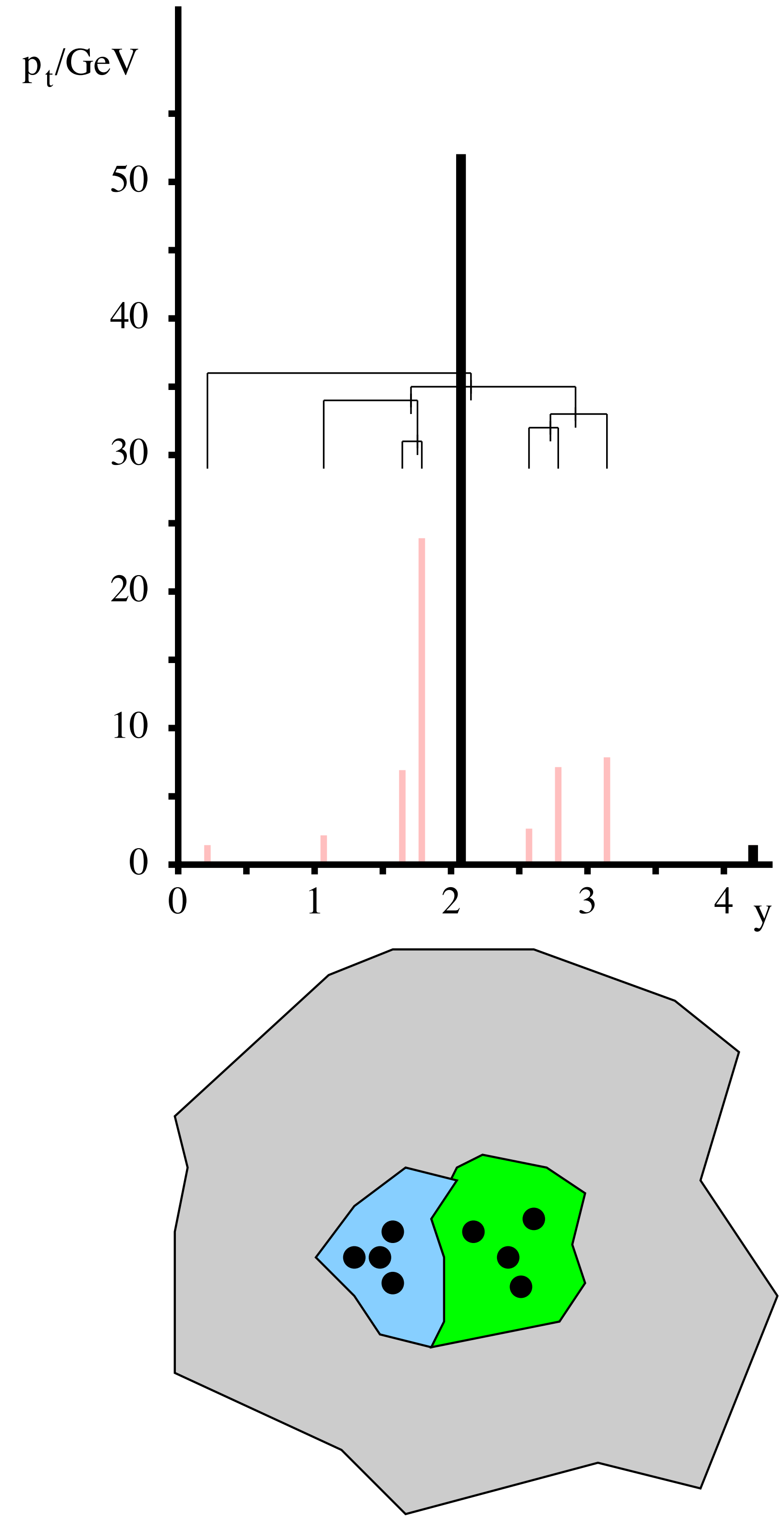
anti-k_t algorithm



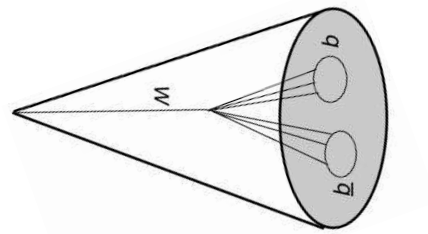
k_t algorithm



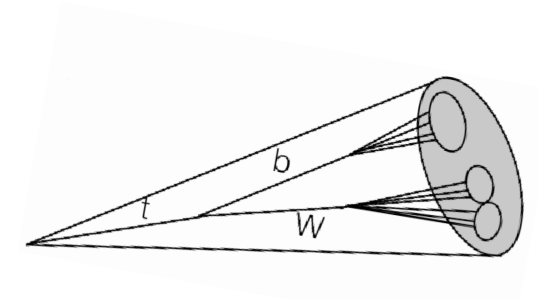
Cambridge/Aachen

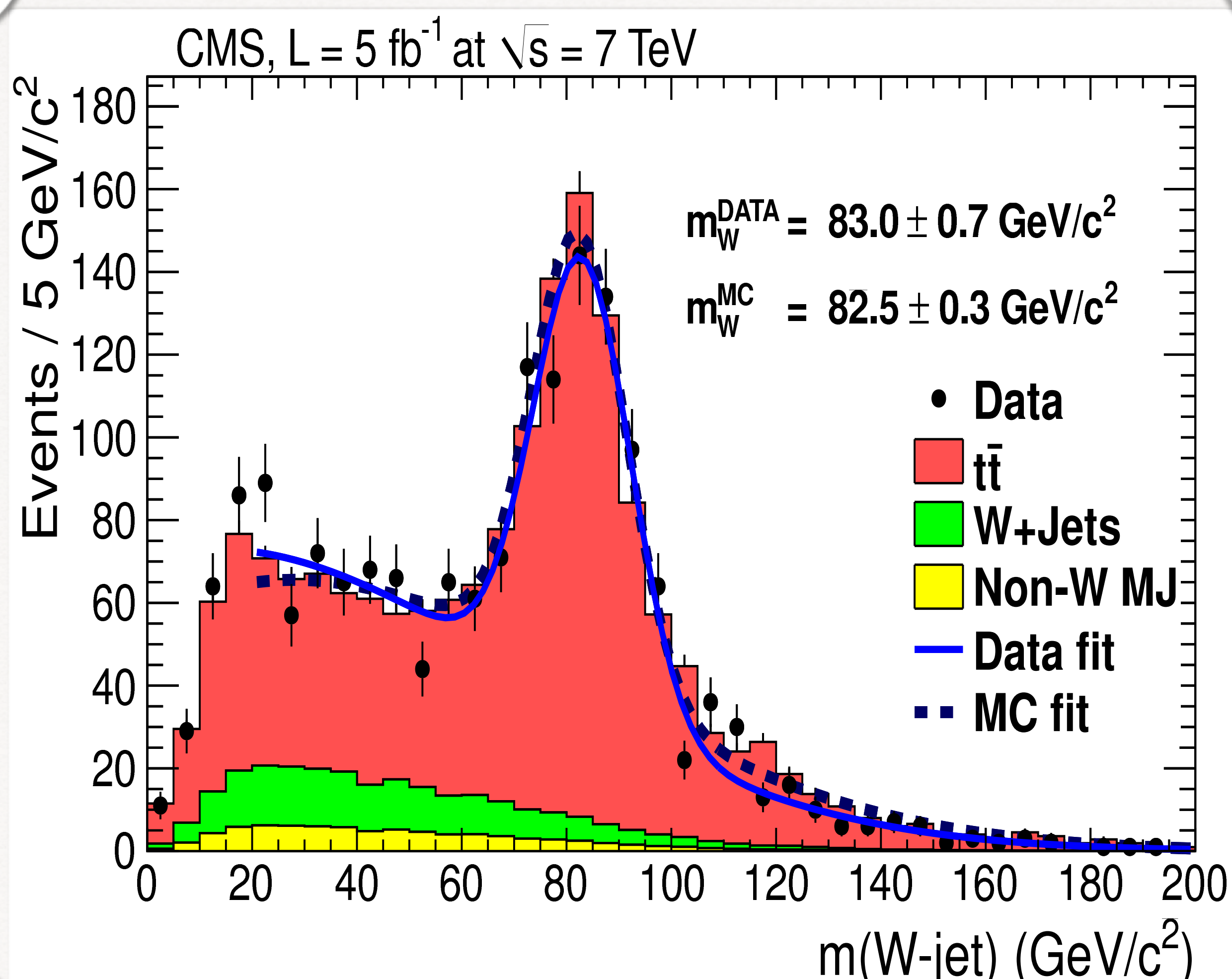


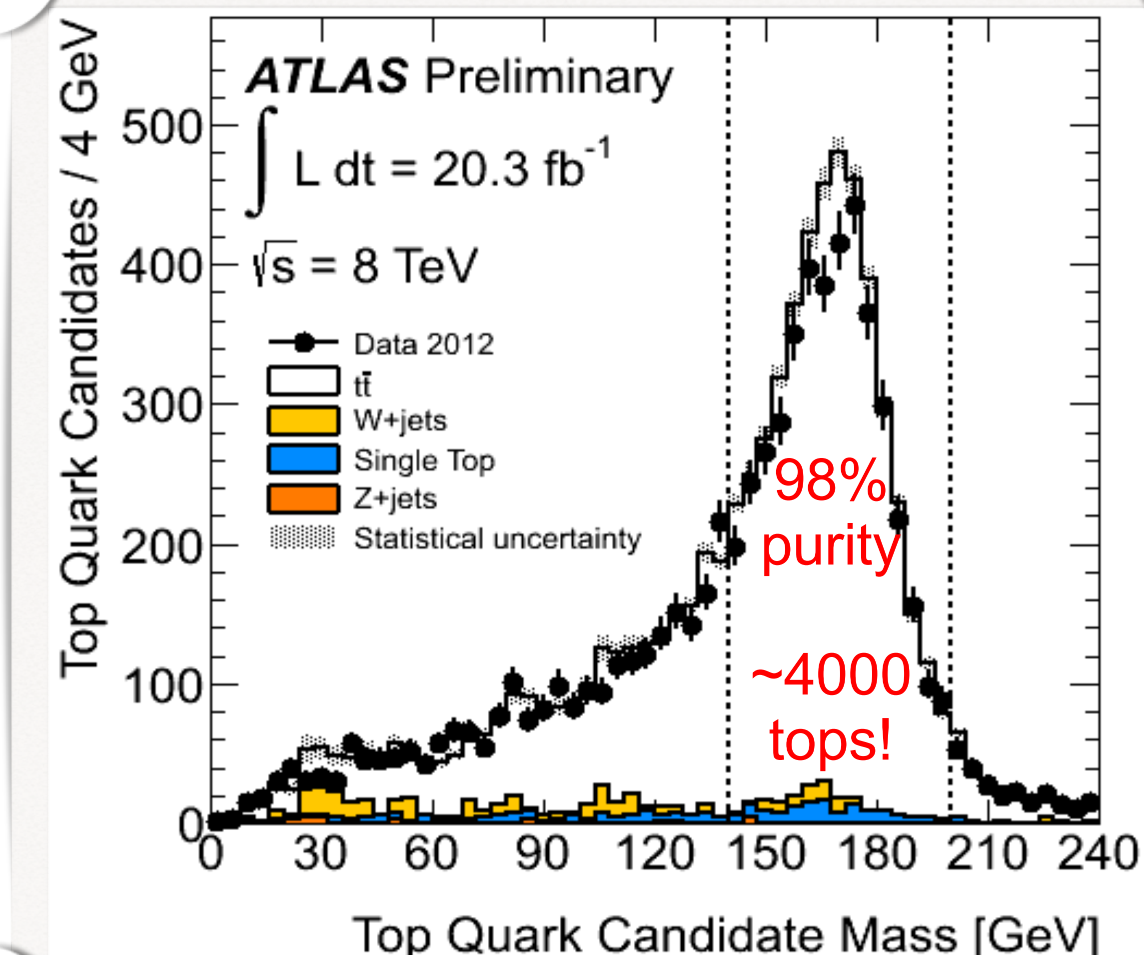
W's in a single jet



tops in a single jet



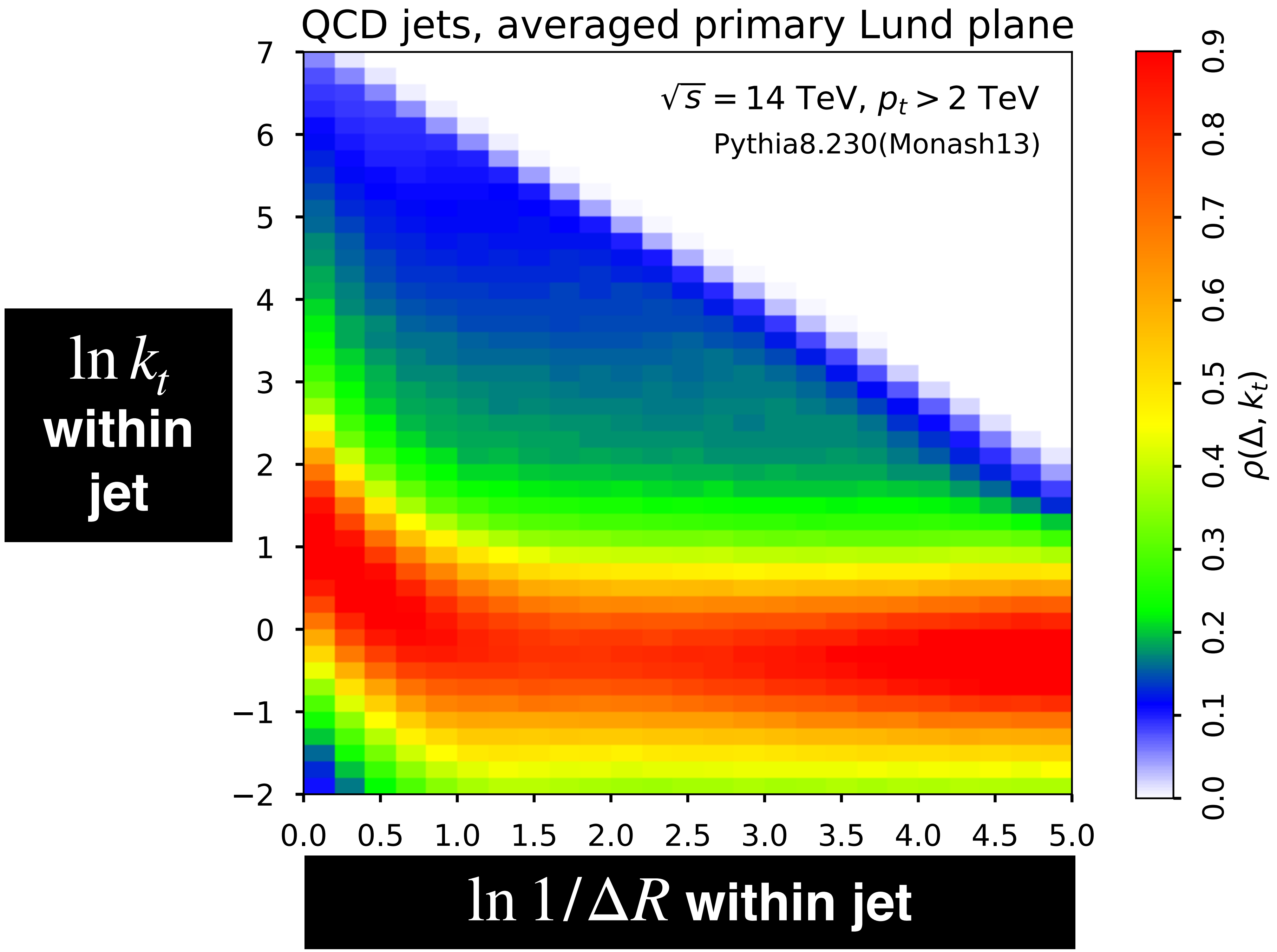




today's jet substructure tools

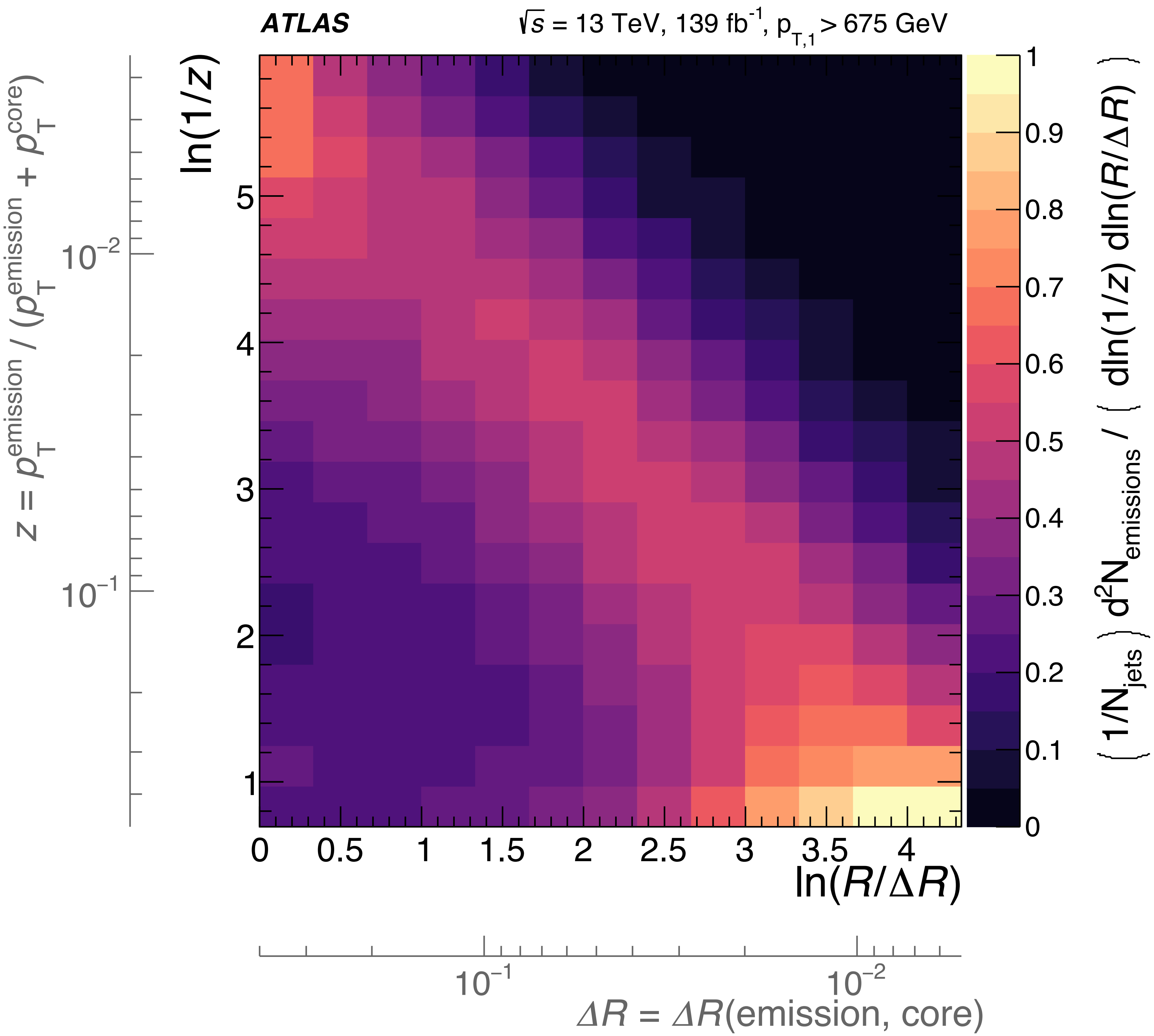
- **SoftDrop:** uses the same key ideas of C/A declustering, but with better theoretical properties and more flexibility in phasespace
- **Subjettiness / energy-energy-correlations / energy-flow polynomials / Lund Plane structure:** all try to measure the energy flow around the core n-prong structure of a jet (e.g. 2-prong for Higgs decay)
- **Machine learning:** jet substructure is one of the most dynamic playgrounds for ML, with large gains to be had in pulling out all info from jets

density of intrajet emissions in QCD jets



Dreyer, GPS & Soyez, 1807.04758; Lifson, GPS & Soyez, 2007.06578

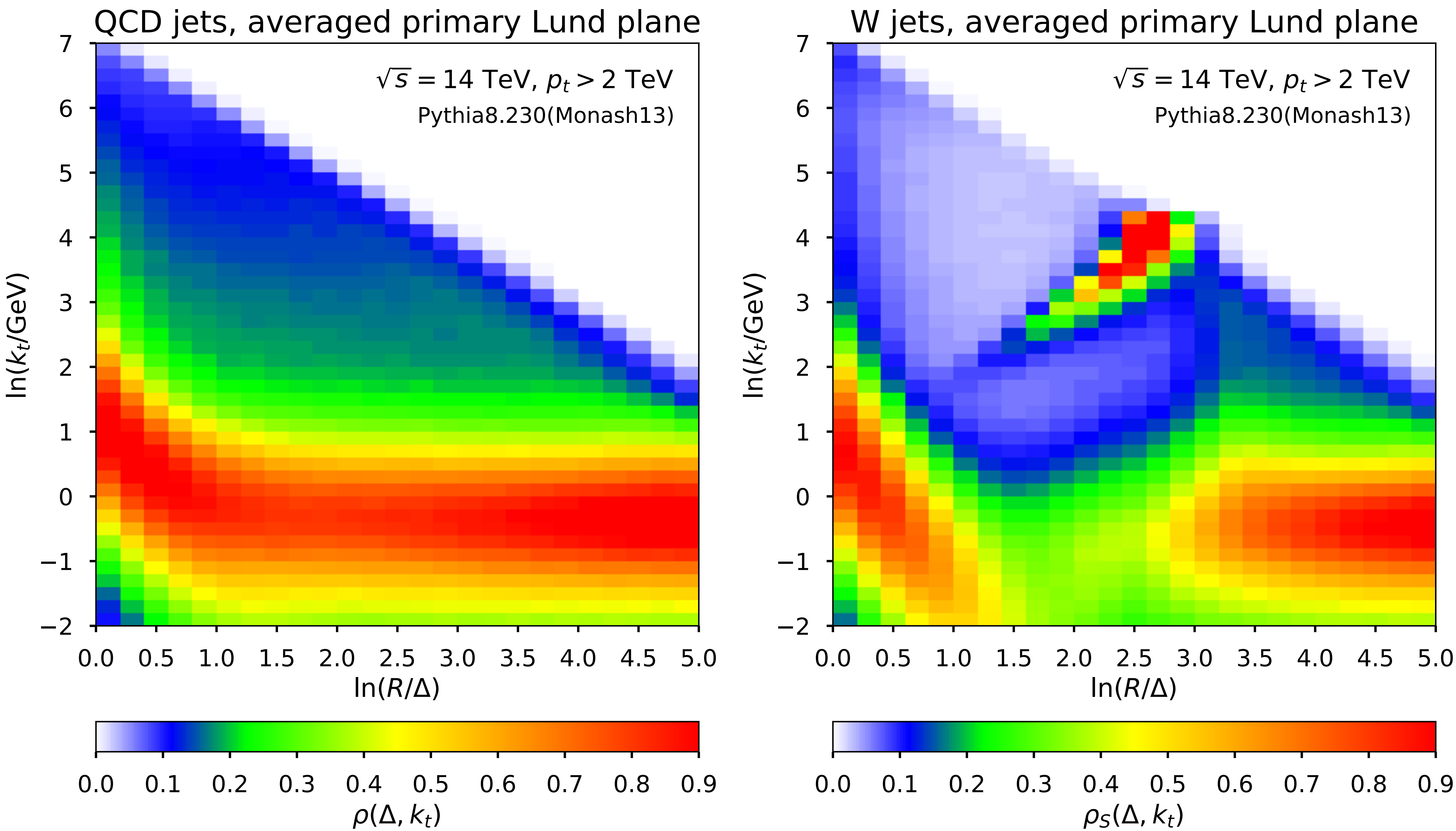
ATLAS measurement of Lund jet plane



ATLAS
2004.03540

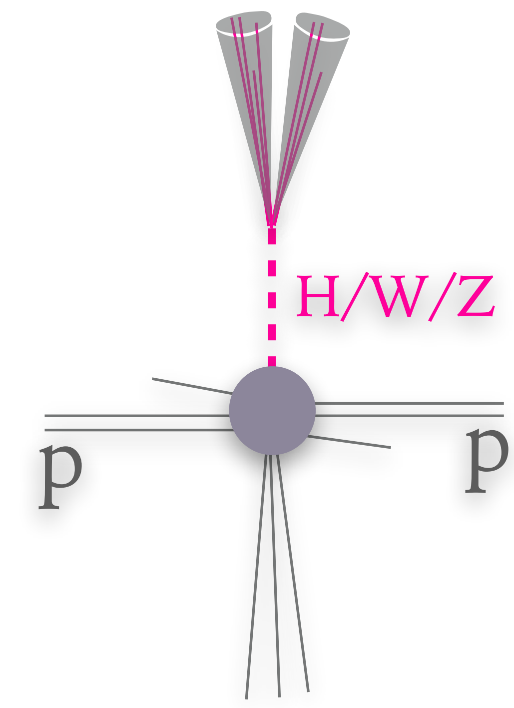
NB: vertical axis is $\ln z$ rather than $\ln k_t$

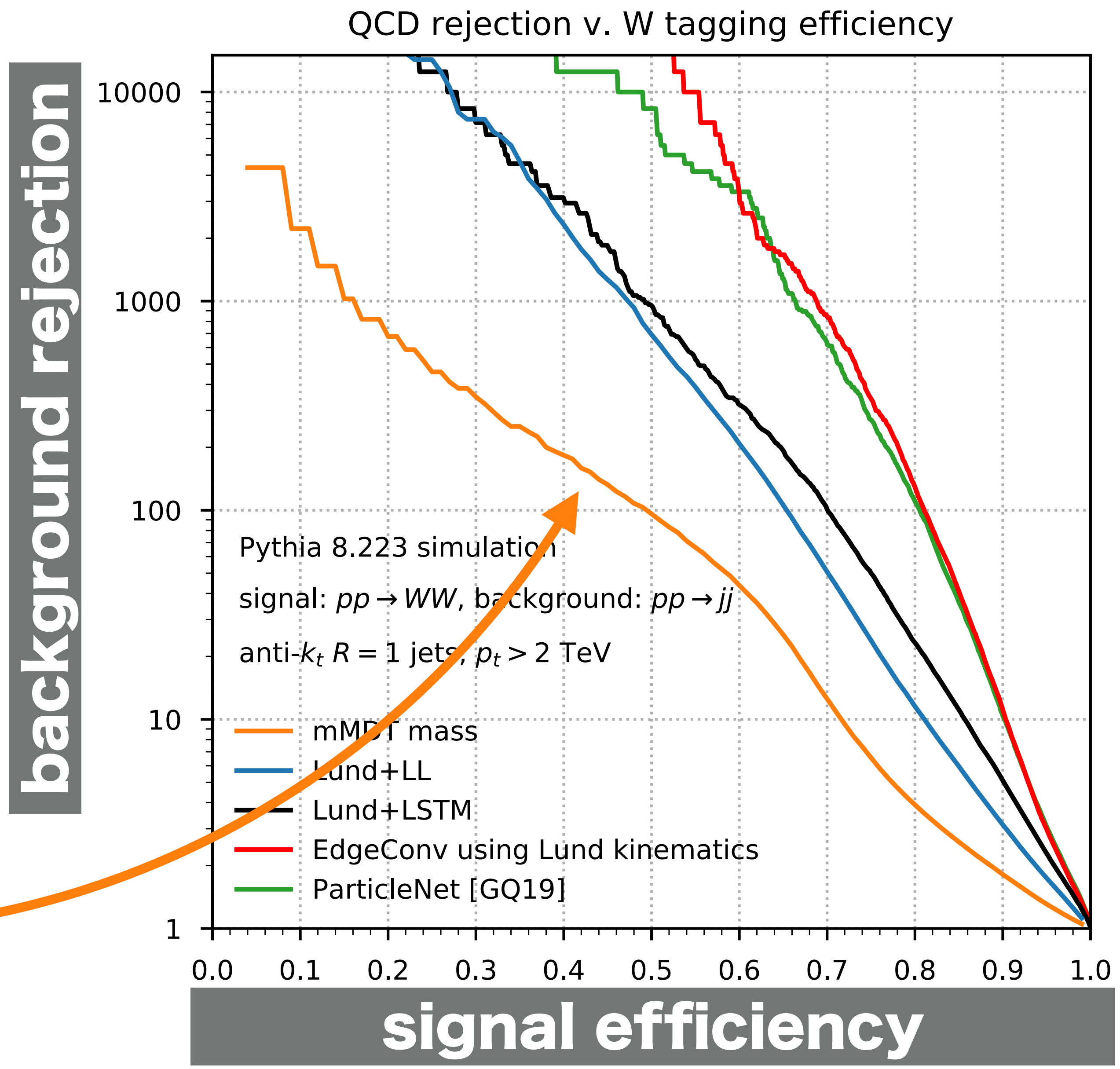
intrajet energy flow for QCD jets & W jets



using full jet/event information for H/W/Z-boson

F. Dreyer & H. Qu, 2012.08526





QCD rejection with just jet mass (SD/mMDT) i.e. 2008 tools & their 2013/14 descendants

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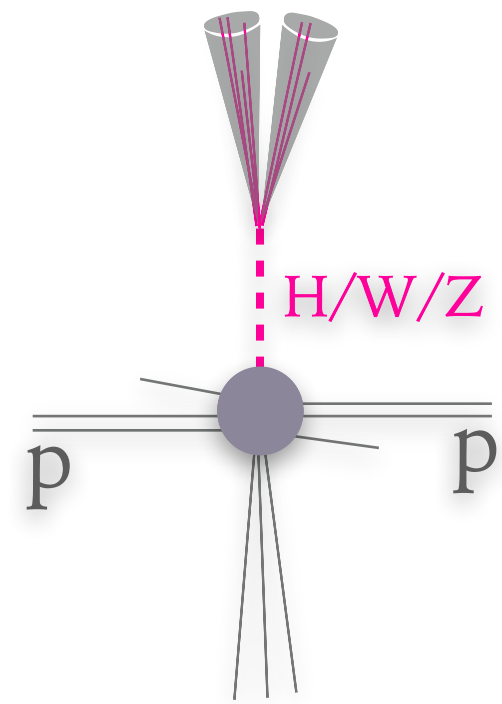
F. Dreyer & H. Qu, 2012.08526



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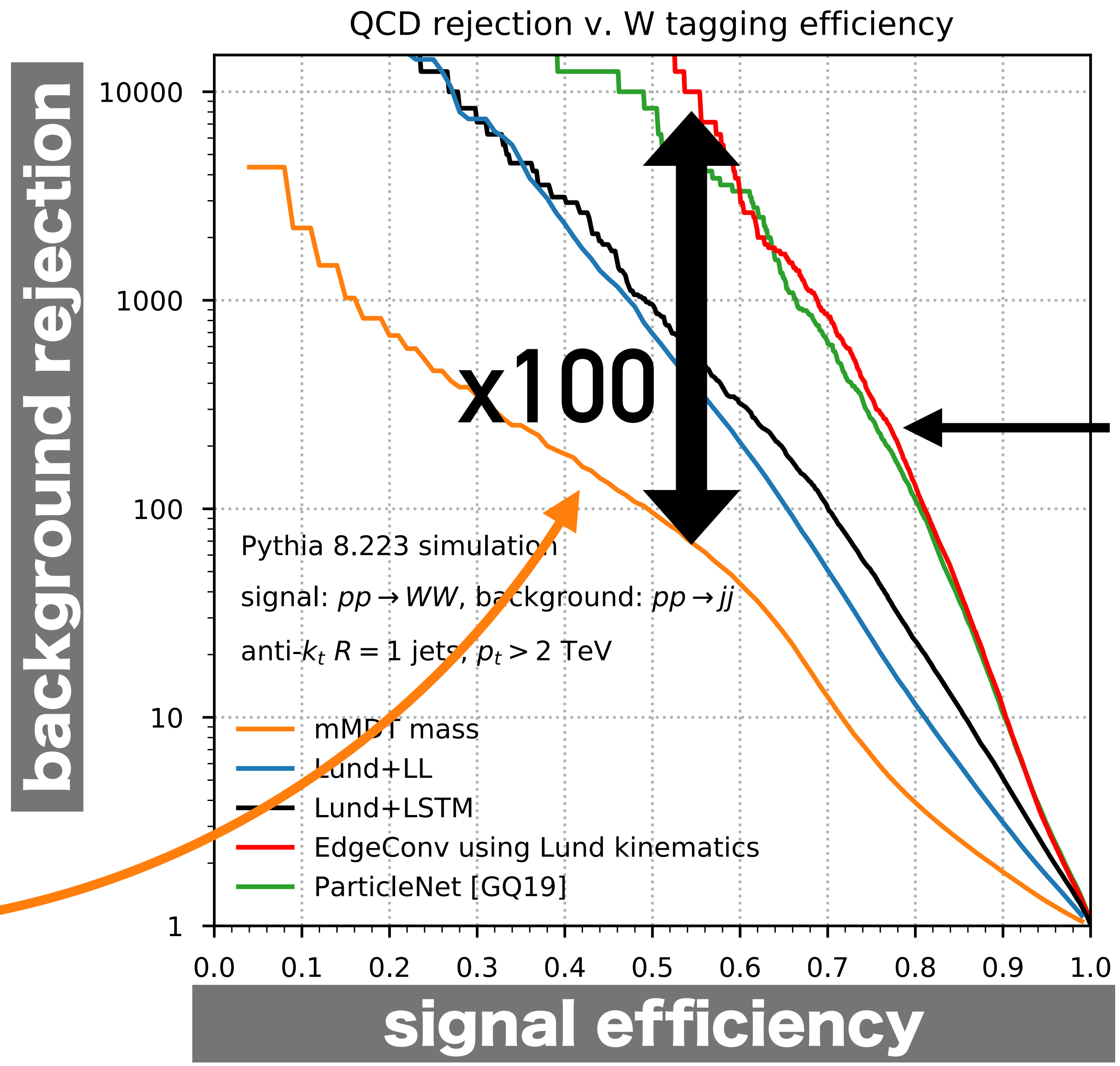


QCD rejection with use of full jet substructure (2019 tools)

100x better

First started to be exploited by Thaler & Van Tilburg with "N-subjettiness" (2010/11)

Take-home messages

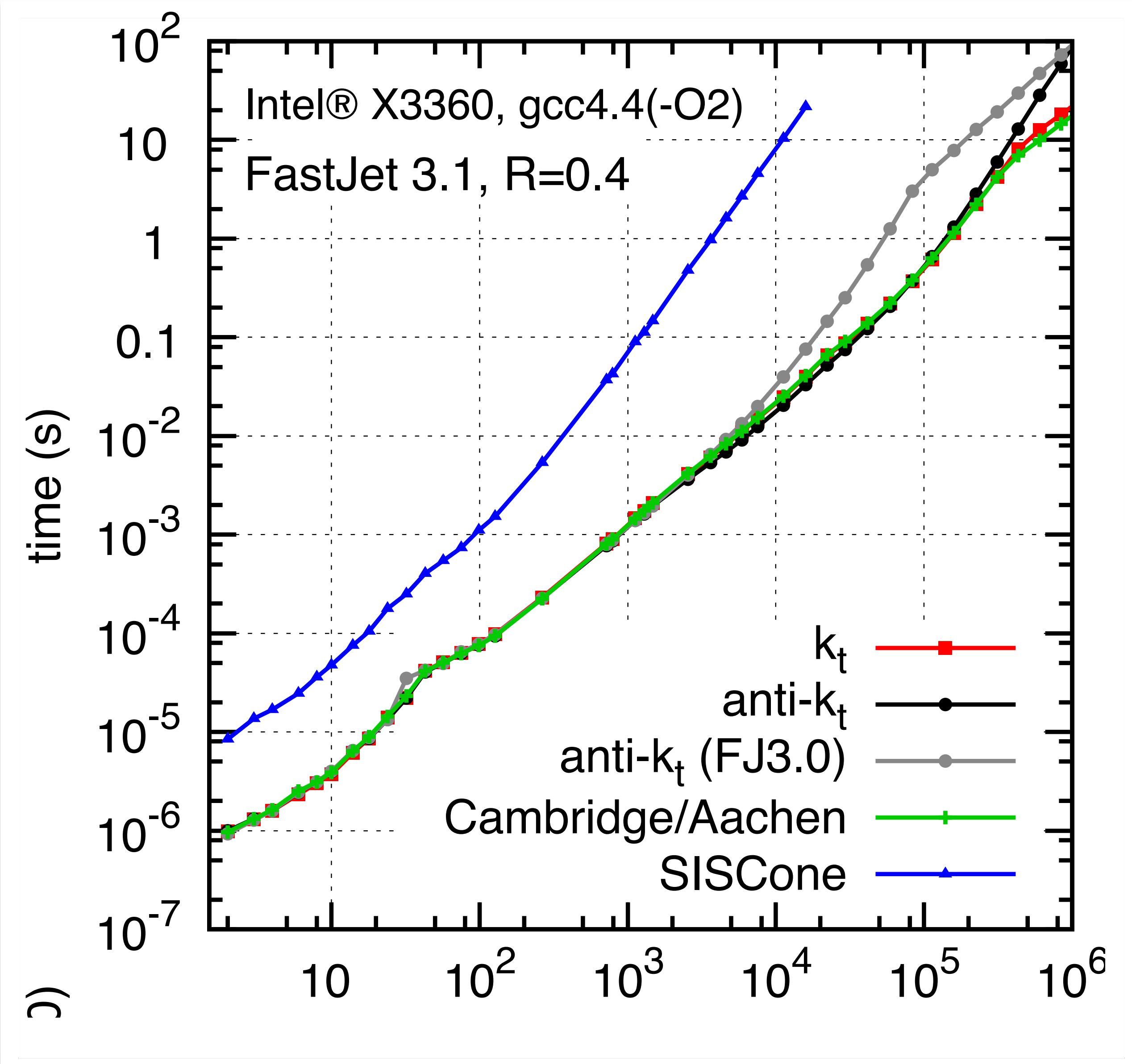
- Jets are a consequences of the soft & collinear enhancements of gluon emission (even at small coupling), followed by hadronisation
- There are myriad approaches to jet finding
- For applications with a single moderately hard scale (e.g. ttbar), anti-kt, R=0.4, with a p_t cut of a few tens of GeV is often a good default
- For problems with multiple hard scales (e.g. highly boosted top / W / H / etc.) one needs to look at events on multiple angular scales: jet substructure

References with more info

- Towards Jetography, *GPS*, 0906.1833
- Jet Substructure at the Large Hadron Collider: A Review of Recent Advances in Theory and Machine Learning, *Larkoski, Moult and Nachman*, 1709.04464
- Jet Substructure at the Large Hadron Collider: Experimental Review, *L. Asquith et al.*, 1803.06991
- Looking inside jets: an introduction to jet substructure and boosted-object phenomenology, M*arzani, Soyez and Spannowsky*, 1901.10342

EXTRAS

Time to cluster N particles in FastJet



Time to cluster N particles

FJContrib packages

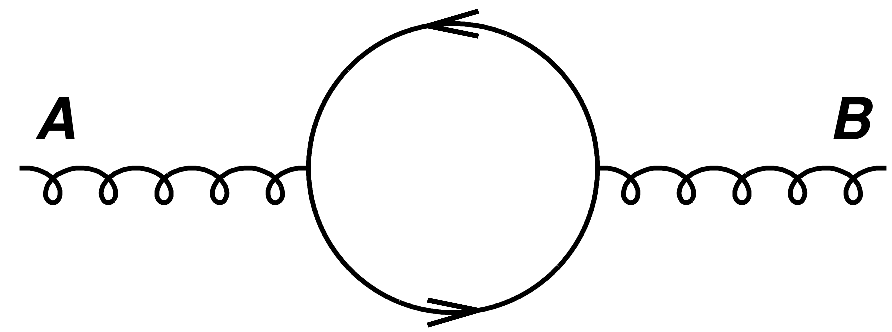
Version 1.045 of FastJet Contrib is distributed with the following packages

Package	Version	Release date	Information
Centauro	1.0.0	2020-08-04	README NEWS
ClusteringVetoPlugin	1.0.0	2015-05-04	README NEWS
ConstituentSubtractor	1.4.5	2020-02-23	README NEWS
EnergyCorrelator	1.3.1	2018-02-10	README NEWS
FlavorCone	1.0.0	2017-09-07	README NEWS
GenericSubtractor	1.3.1	2016-03-30	README NEWS
JetCleanser	1.0.1	2014-08-16	README NEWS
JetFFMoments	1.0.0	2013-02-07	README NEWS
JetsWithoutJets	1.0.0	2014-02-22	README NEWS
LundPlane	1.0.3	2020-02-23	README NEWS
Nsubjettiness	2.2.5	2018-06-06	README NEWS
QCDAwarePlugin	1.0.0	2015-10-08	README NEWS
RecursiveTools	2.0.0	2020-03-03	README NEWS
ScJet	1.1.0	2013-06-03	README NEWS
SoftKiller	1.0.0	2014-08-17	README NEWS
SubjetCounting	1.0.1	2013-09-03	README NEWS
ValenciaPlugin	2.0.2	2018-12-22	README NEWS
VariableR	1.2.1	2016-06-01	README NEWS

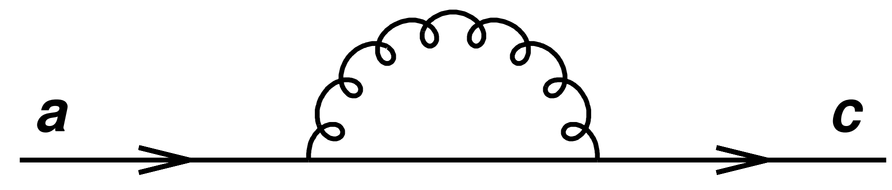
more details on soft emission

Quick guide to colour algebra

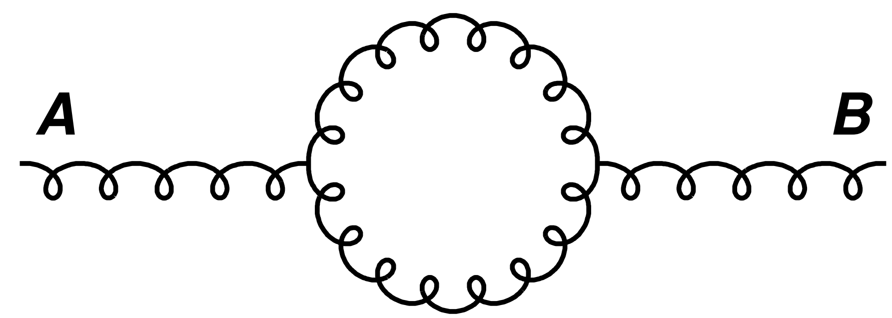
$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \qquad T_R = \tfrac{1}{2}$$



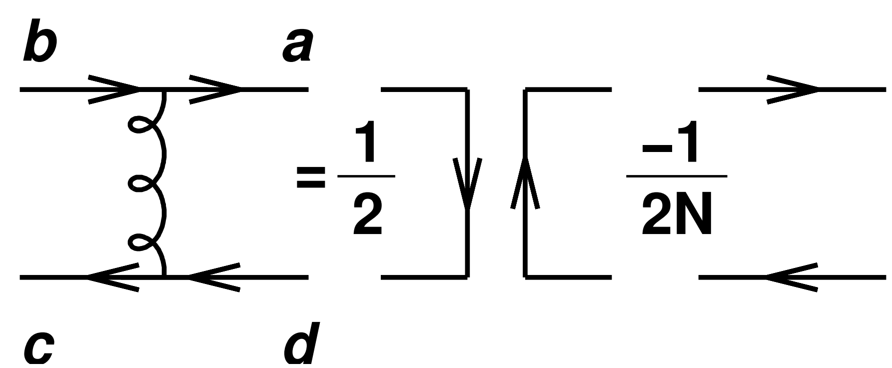
$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \qquad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$



$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \qquad C_A = N_c = 3$$



$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_c}\delta_{ab}\delta_{cd} \quad \text{(Fierz)}$$

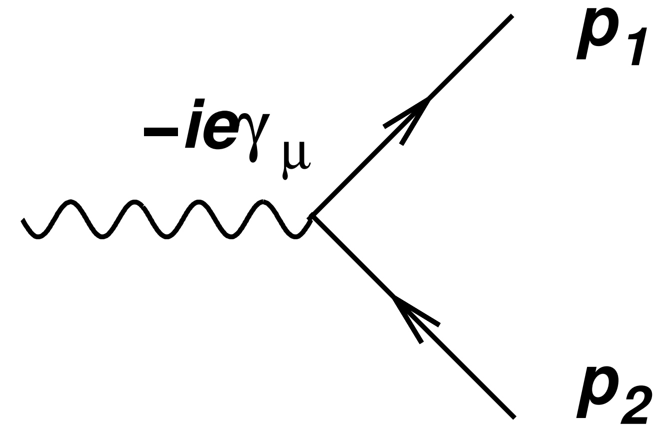


$N_c \equiv$ number of colours $= 3$ for QCD

Soft gluon amplitude

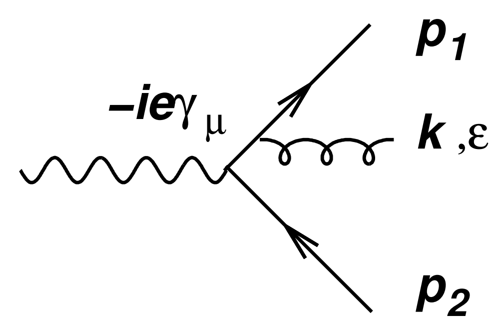
Start with $\gamma^* \to q\bar{q}$:

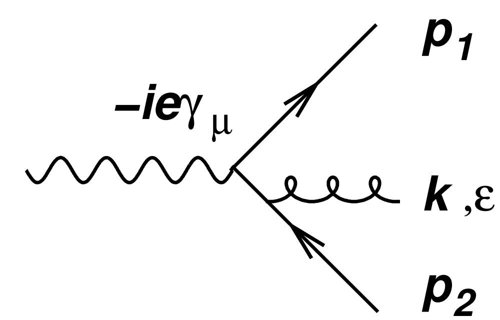
$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2)$$



Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$





Make gluon *soft* $\equiv k \ll p_{1,2}$; ignore terms suppressed by powers of k:

$$\mathcal{M}_{q\bar{q}g} \simeq \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

$\not{p} v(p) = 0,$
$\not{p}\not{k} + \not{k}\not{p} = 2p.k$

Soft gluon amplitude

Start with $\gamma^* \to q\bar{q}$:

$$\bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) = -i g_s \bar{u}(p_1) \not{\epsilon} \frac{\not{p}_1 + \not{k}}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\not{A}\not{B} = 2A.B - \not{B}\not{A}$:

$$= -i g_s \bar{u}(p_1)[2\epsilon.(p_1+k) - (\not{p}_1 + \not{k})\not{\epsilon}] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\bar{u}(p_1)\not{p}_1 = 0$ and $k \ll p_1$ (p_1, k massless)

$$\simeq -i g_s \bar{u}(p_1)[2\epsilon.p_1] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

$$= -i g_s \frac{p_1.\epsilon}{p_1.k} \underbrace{\bar{u}(p_1) e_q \gamma_\mu t^A v(p_2)}_{\text{pure QED spinor structure}}$$

Squared amplitude

$$|M^2_{q\bar{q}g}| \simeq \sum_{A,\text{pol}} \left| \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

$$= -|M^2_{q\bar{q}}| C_F g_s^2 \left(\frac{p_1}{p_1.k} - \frac{p_2}{p_2.k} \right)^2 = |M^2_{q\bar{q}}| C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

Include phase space:

$$d\Phi_{q\bar{q}g} |M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}} |M^2_{q\bar{q}}|) \frac{d^3\vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

Note property of factorisation into hard $q\bar{q}$ piece and soft-gluon emission piece, $d\mathcal{S}$.

$$\boxed{d\mathcal{S} = E dE\, d\cos\theta\, \frac{d\phi}{2\pi} \cdot \frac{2\alpha_s C_F}{\pi} \frac{2p_1.p_2}{(2p_1.k)(2p_2.k)}}$$

$\theta \equiv \theta_{p_1 k}$
$\phi =$ azimuth