Matrix combination of BFKL and DGLAP

Gavin Salam

LPTHE, Universities of Paris VI and VII and CNRS

Work with M. Ciafaloni, D. Colferai and A. Stasto
1998–2007 and especially arXiv:0707.1453 [hep-ph]

BNL, 6 December 2007

Introduction

This talk is a progress report on a long-term project to put together *DGLAP* and the linear regime of *BFKL* evolution, including higher order and running-coupling corrections.

Main groups active:

- Altarelli, Ball, Forte (+ Falgari, Marzano) aka ABF
- Ciafaloni, Colferai, GPS, Staśto aka CCSS

\+ Thorne & White

DGLAP, BFKL (fixed coupling)

DGLAP

Integro(x)–differential(Q^2) eq^n for *integrated* gluon dist., g:

$$\frac{dg(x,Q^2)}{d\ln Q^2} = \int \frac{dz}{z} P_{gg}(z) g\left(\frac{x}{z}, Q^2\right)$$

BFKL

Integro(k)–differential(x) eq^n for *unintegrated* gluon dist., G:

$$\frac{dG(x,k^2)}{d\ln 1/x} = \int \frac{dk'^2}{k'^2} K(k/k') G(x, k'^2)$$

k, Q are transverse scales; x is longitudinal mom. fraction

$xg(x,Q^2) = \int^Q d^2k\, G(x,k^2)$

Both DGLAP and BFKL relate $\perp$ structure to long. structure:

- given long. struct. DGLAP gives you $\perp$ struct. evolution
- given $\perp$ struct. BFKL gives you long. struct. evolution

When calculated at all orders they must encode the same physics.

Inevitable that one contaminated by other at fixed order

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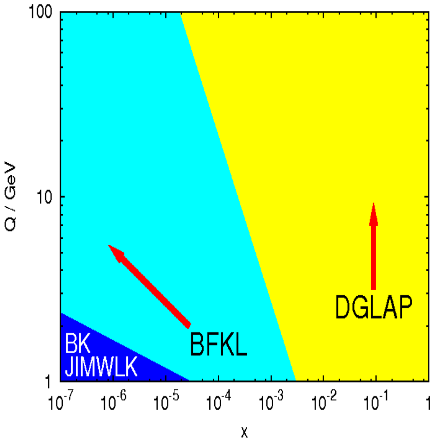
Motivation

When looking at proton structure we can establish different evolution regimes (NB: picture for proton):



But:

- Regions of validity not clearly delimited
- Higher orders of DGLAP contaminated by leading BFKL:

$$P_{gg}(x) \simeq \frac{\bar{\alpha}_s}{x} + \bar{\alpha}_s^4 \frac{\zeta(3)}{3} \frac{\ln^3 x}{x} + \dots$$

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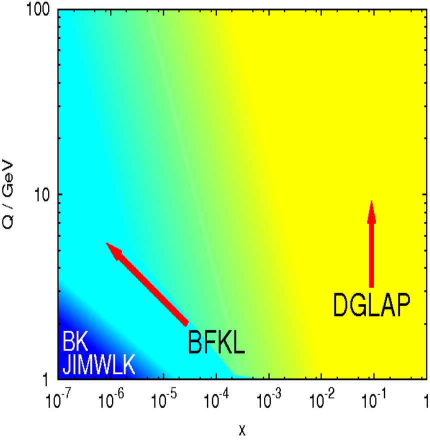

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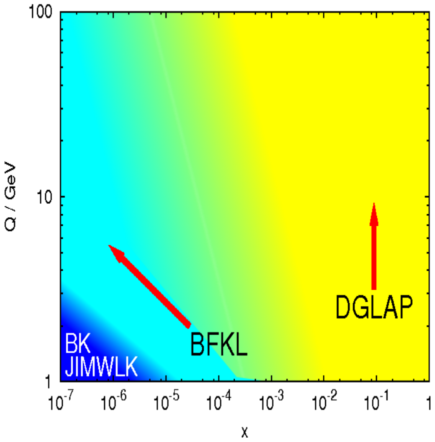

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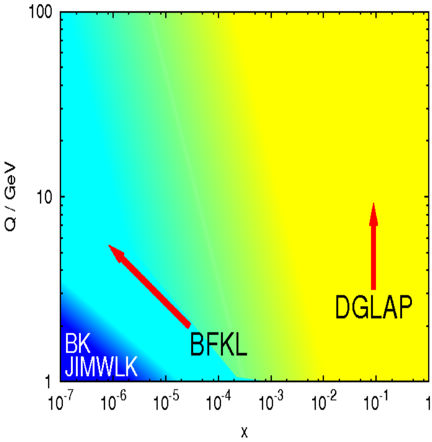

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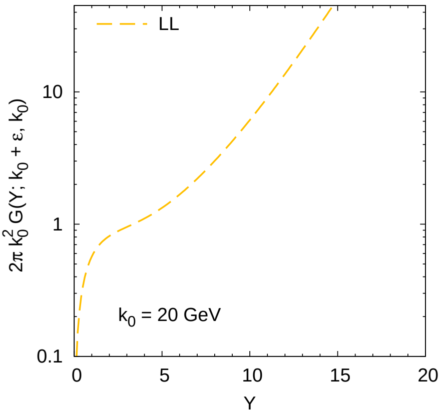
NLL BFKL Green function solution

If DGLAP contaminates BFKL does it matter? Can we not just take the perturbative expansion? Try LL, then NLL BFKL.



Choices that formally only affect NNLLx:

- scale of α_s
- 'energy-scale' s_0 ($Y = \ln s/s_0$).

lead to completely different answers →

Source of instability is presence in NLL BFKL of a truncated subset of DGLAP. Only way to get stability is to include full DGLAP.

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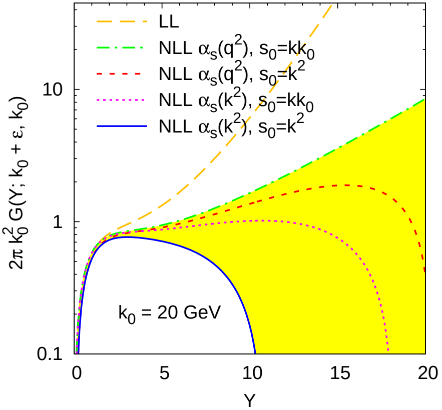

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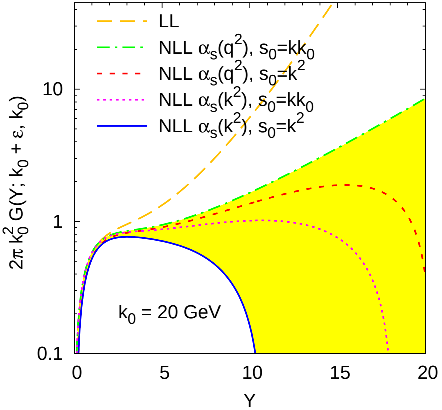

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Perturbative structure of DGLAP P_{gg}

- Small-x gluon splitting function has logarithmic enhancements:

$$xP_{gg}(x) = \sum_{n=1} \alpha_s^n \ln^{n-1}\frac{1}{x}$$

$$+ \sum_{n=2} \alpha_s^n \ln^{n-2}\frac{1}{x} + \dots$$

- NNLO (α_s^3): first small-x enhancement in gluon splitting function.

Leading Logs (LLx)

$$\bar{\alpha}_s + \frac{\zeta(3)}{3}\bar{\alpha}_s^4 \ln^3\frac{1}{x} + \frac{\zeta(5)}{60}\bar{\alpha}_s^6 \ln^5\frac{1}{x} + \cdots$$

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$$A_{20}\bar{\alpha}_s^2 + A_{31}\bar{\alpha}_s^3 \ln\frac{1}{x} + A_{42}\bar{\alpha}_s^4 \ln^3\frac{1}{x} + \dots$$

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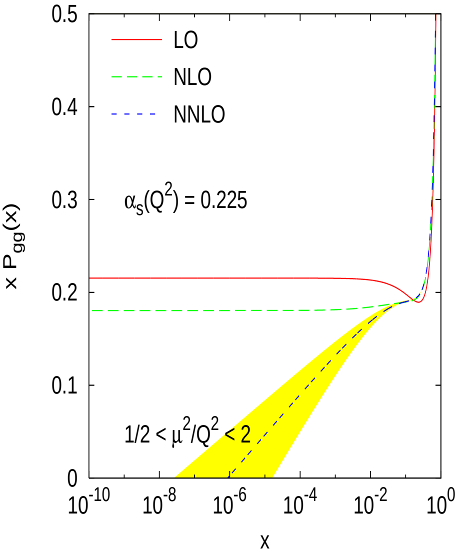


Moch, Vermaseren & Vogt

Merging BFKL & DGLAP

Long history of work on merging leading BFKL and DGLAP.

CCFM '88; Lund group ∼ '95; Durham-Cracow group ∼'95;

Two approaches have been used in order to combine BFKL and DGLAP *including higher orders:*

- Establish all-order relation (*duality relation*) between splitting functions (DGLAP) and evolution kernel (BFKL). Use that to simultaneously construct splitting functions consistent with BFKL kernel and vice-versa.

  Altarelli, Ball & Forte '99–

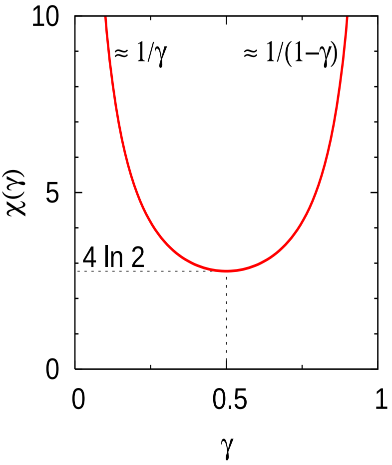
- Establish a more general equation that embodies both BFKL and DGLAP (*double-integral equation*):

$$G(x,k^2) = G_0(x,k^2) + \int dz \int dk'^2 \frac{dk'^2}{k'^2} K(z,k,k') G(x/z,k'^2)$$

  From that, deduce *effective* splitting function and BFKL kernel.

  Ciafaloni, Colferai, GPS & Staśto, '98-

Characteristic function



Eigenvalues of BFKL kernel:

$$\mathcal{K} \otimes (k^2)^\gamma = \bar{\alpha}_s \chi(\gamma) \cdot (k^2)^\gamma$$

$\chi(\gamma)$ is *characteristic function*

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$\rightarrow$ high energy evolution, $\sigma \sim e^{\bar{\alpha}_s \chi(\gamma) Y}$.

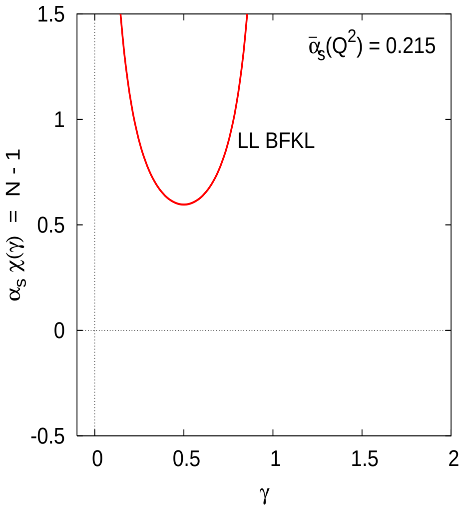
- dominant part at high energies is *minimum* (only stable solution)

$$\sigma \sim e^{4 \ln 2 \bar{\alpha}_s Y} \sim e^{0.5 Y}$$

$\alpha_s \simeq 0.2$

- pole $(1/\gamma)$ corresponds to $\perp$ DGLAP logarithms $\rightarrow$ DL terms $\alpha_s Y \ln Q^2$

Building up the kernel. . .



Examine $\bar{\alpha}_s\chi(\gamma)$

minimum = BFKL power

$$\chi(\gamma) = \underbrace{\chi_0(\gamma)}_{LL} + \underbrace{\bar{\alpha}_s\chi_1(\gamma)}_{NLL} + \ldots$$

- NLL terms *pathologically large.*
  minimum → max. (unstable)
  oscillating X-sctns, . . .
- Culprit: ⊥ DGLAP logs
  $$\frac{\bar{\alpha}_s}{\gamma} - \frac{11}{12}\frac{\bar{\alpha}_s^2}{\gamma^2} + \ldots$$
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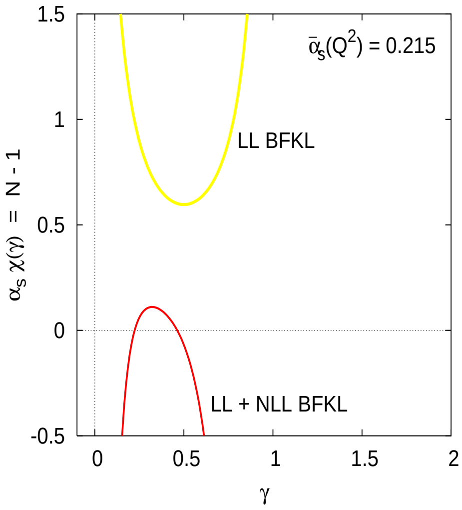

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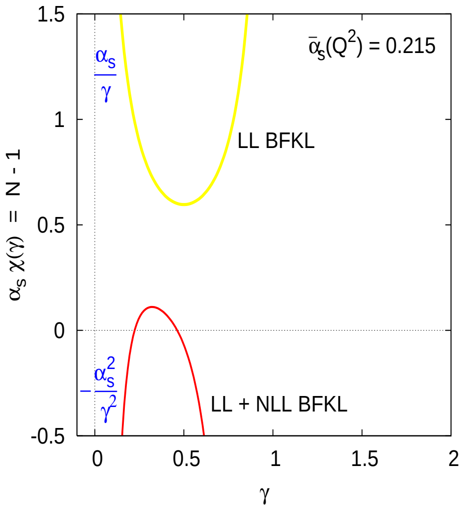

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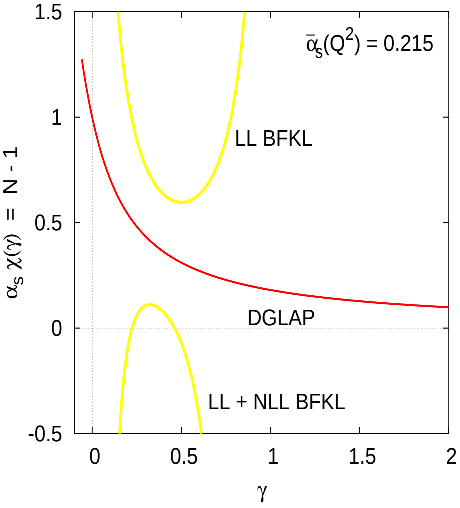

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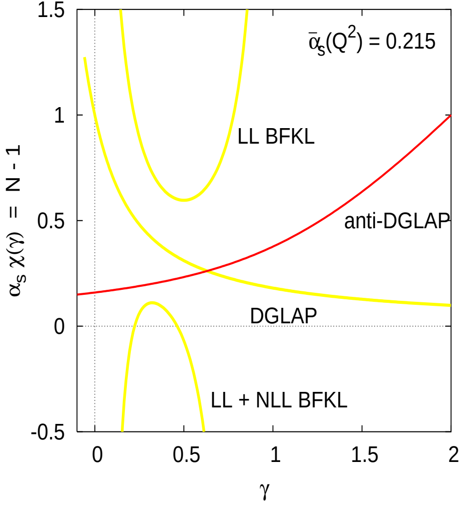

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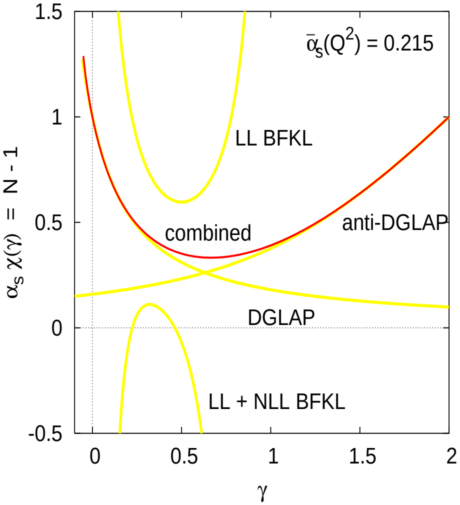

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Assemble all constraints:

stable, sensible kernel

Ciafaloni, Colferai, GPS & Staśto '98–'03;
Altarelli, Ball & Forte; '99–'05

NB: cf. strong coupling limits with same fixed points at $\gamma = 0, 2$

Brower, Polchinski, Strassler & Tan '06
Stasto '07

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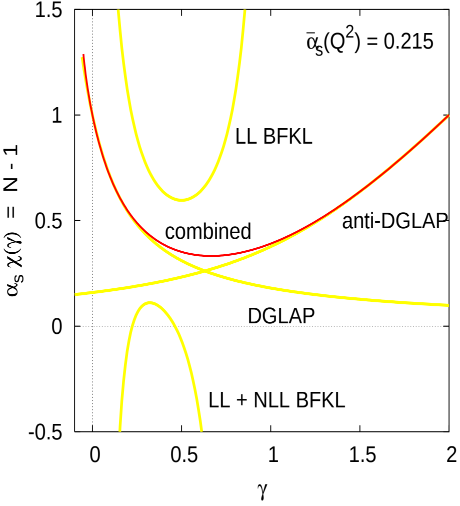

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Pure glue case, LLx+LO

Write Kernel as power series in α_s: $K = \sum_{n=0} \hat{\alpha}^n K_n$ $\qquad \hat{\alpha} = \alpha_s/2\pi$

First order (*LLx-LO*) has two parts:

$$K_0(\gamma,\omega) = \underbrace{\frac{2C_A}{\omega}\chi_0^\omega(\gamma)}_{\text{BFKL (LL}x\text{)}} + \underbrace{\left[\Gamma_{gg,0}(\omega) - \frac{2C_A}{\omega}\right]\chi_c^\omega(\gamma)}_{\text{finite-}x\text{ DGLAP (LO)}}$$

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BFKL piece has usual transverse structure with *kinematic constraint*

$$\chi_0^\omega(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1+\omega-\gamma)$$

Note symmetry $\gamma \leftrightarrow 1-\gamma+\omega$

Multiplied by $\alpha_s(q^2)$, $\vec{q} = \vec{k} - \vec{k}'$

DGLAP remainder piece has a *collinear kernel*:

$$\chi_c^\omega(\gamma) = \frac{1}{\gamma} + \frac{1}{1+\omega-\gamma}$$

Multiplied by $\alpha_s(k_>^2)$

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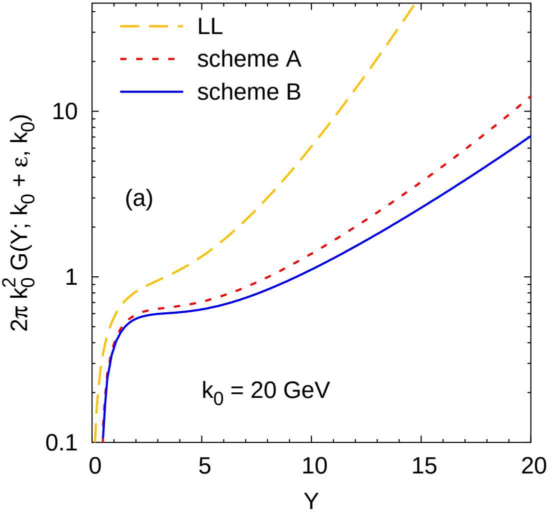
Next order (*NLx-NLO*) also has two parts:

$$K_1(\gamma,\omega) = \frac{(2C_A)^2}{\omega}\,\tilde{\chi}_1^\omega(\gamma) + \tilde{\Gamma}_{gg,1}(\omega)\,\chi_c^\omega(\gamma)$$

with $\tilde{\chi}_1$ and $\tilde{\Gamma}_{gg,1}(\omega)$ adjusted so as to reproduce NLx BFKL and NLO DGLAP.

Green fn. from improved kernel



First tried in '03, without NLO DGLAP piece.

NLx-LO

Two schemes, to estimate degree of stability

- scheme A violates mom. sum-rule at $\mathcal{O}\left(\alpha_s^2\right)$
- scheme B satisfies it at all orders

Solve double-integral eq^n with each.

Different schemes → similar results

cf. pure NLO

Green function ⇒ effective P_{gg}

Construct a gluon density from Green function (take $k \gg k_0$):

$$xg(x, Q^2) \equiv \int^{Q} d^2k \; G^{(\nu_0 = k^2)}(\ln 1/x, k, k_0)$$

Numerically solve equation for effective splitting function, $P_{gg,\mathrm{eff}}(z, Q^2)$:

$$\frac{dg(x, Q^2)}{d \ln Q^2} = \int \frac{dz}{z} P_{gg,\mathrm{eff}}(z, Q^2)\, g\left(\frac{x}{z}, Q^2\right)$$

Factorisation

- Splitting function:

  red paths

- Green function:

  all paths

Green function $\Rightarrow$ effective P_{gg}

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$$xg(x, Q^2) \equiv \int^Q d^2k \; G^{(\nu_0 = k^2)}(\ln 1/x, k, k_0)$$

Numerically solve equation for effective splitting function, $P_{gg,\mathrm{eff}}(z, Q^2)$:

$$\frac{dg(x, Q^2)}{d \ln Q^2} = \int \frac{dz}{z} P_{gg,\mathrm{eff}}(z, Q^2) \, g\left(\frac{x}{z}, Q^2\right)$$

Factorisation

- Splitting function:

  red paths

- Green function:

  all paths

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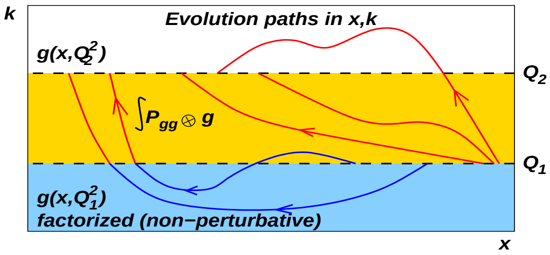
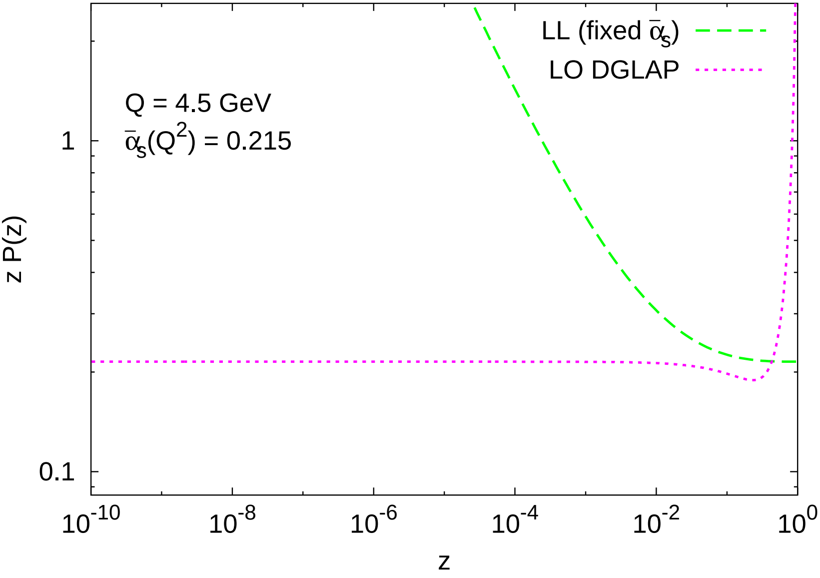


Full $P_{gg}(z)$ splitting fn



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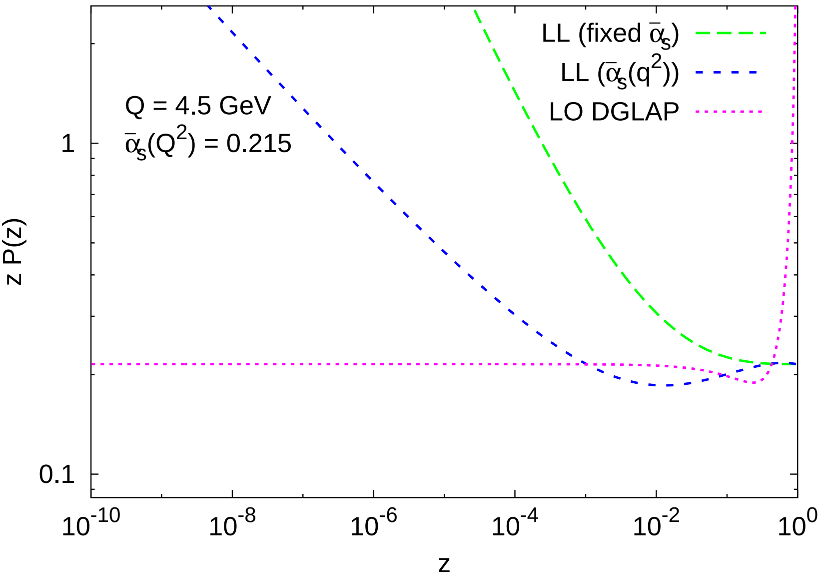

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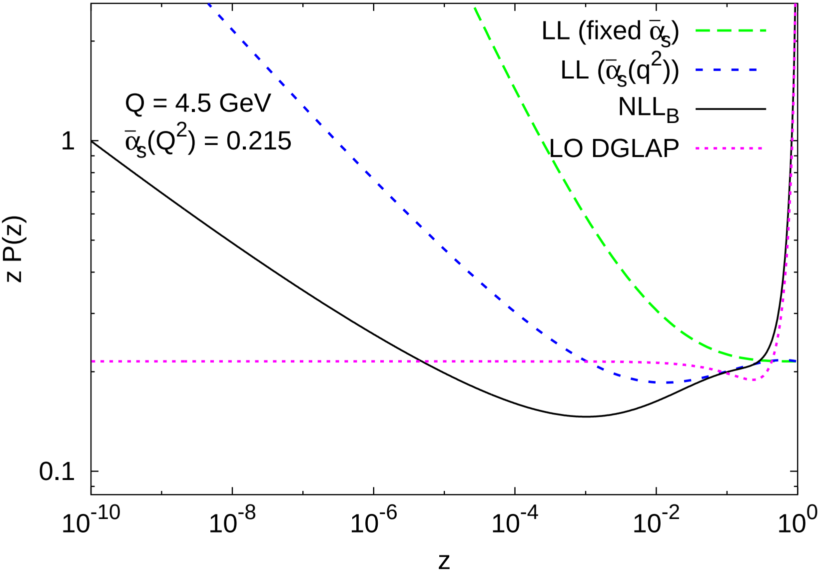
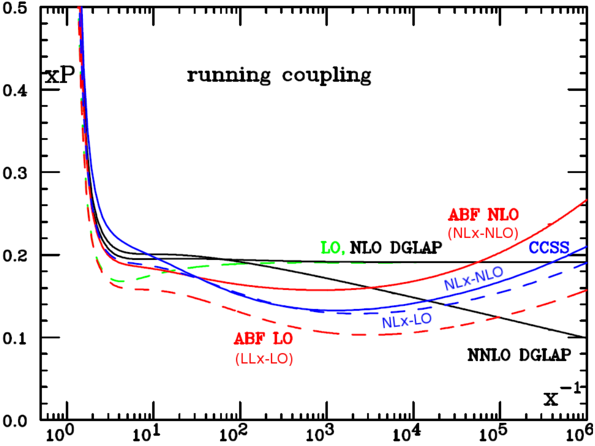


One channel P_{gg}: ABF v. CCSS



Altarelli, Ball & Forte have also calculated effective P_{gg}:

- similar physical ingredients
- completely different 'implementation'

Main features similar between CCSS & ABF.

In particular splitting-fn has **dip** at $x \sim 10^{-3}$.

The dip

	LLx	NLLx	NNLLx	...
α_s	x	–	–	
α_s^2	0	n_f	–	
α_s^3	0	x	x	
α_s^4	x	x	x	
α_s^5	0	x	x	
⋮				

const.
ln 1/x
$\ln^2 1/x$
$\ln^3 1/x$

At moderately small x, first terms with x-dependence are

$$-1.54\,\bar{\alpha}_s^3 \ln\frac{1}{x} + 0.401\,\bar{\alpha}_s^4 \ln^3\frac{1}{x}$$

Minimum when

$$\alpha_s \ln^2 x \sim 1 \quad \equiv \quad \ln\frac{1}{x} \sim \frac{1}{\sqrt{\alpha_s}}$$

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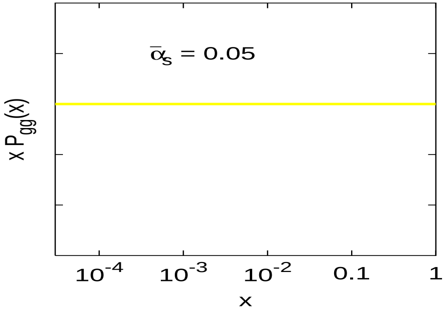

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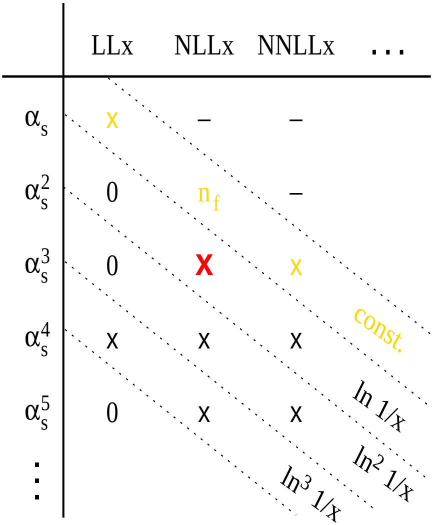

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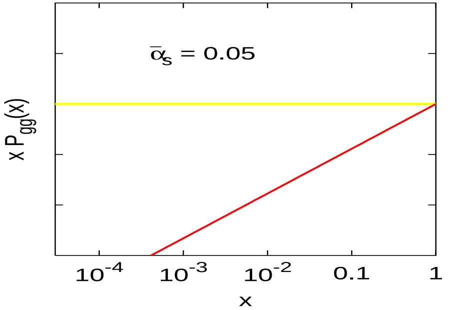

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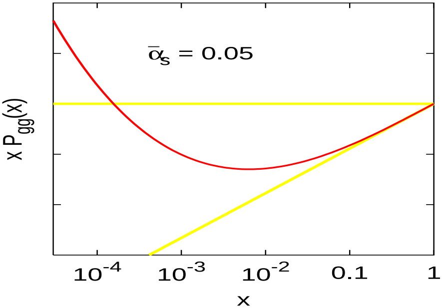

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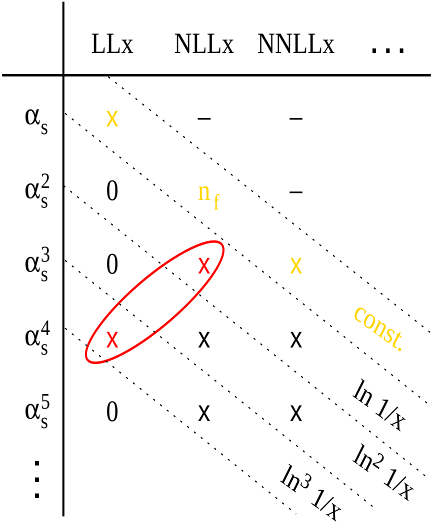

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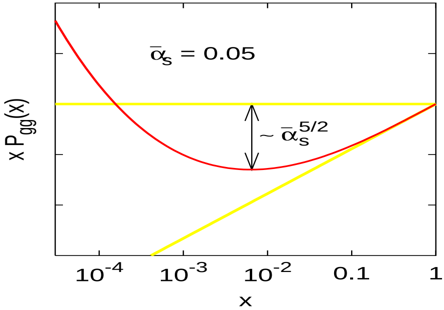

Two channels

BFKL is naturally single-channel Only gluon production has $1/x$ divergence

DGLAP is multi-channel Quarks and gluons both have collinear divergences

So far we had *ignored the multi-channel aspect,* for simplicity. But:

- If we are to use small-x resummed splitting functions, we need the whole singlet matrix
- Including quarks in evolution may provide a convenient way of resumming collinear logs in impact factors

Generalise double-integral eq^n to two channels

Add flavour indices to Green function and kernel

$$G_{ab}(x, k^2, k_0^2) = \delta^2(k - k_0)\delta_{ab} + \int dz \int dk'^2 \frac{dk'^2}{k'^2} K_{ac}(z, k, k') G_{cb}(x/z, k'^2, k_0^2)$$

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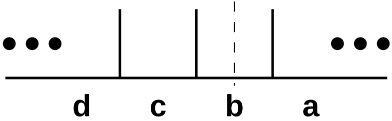
Symmetry and subtleties



Want to encode two strongly ordered collinear limits

DGLAP limit $\quad \ldots \Gamma_{dc} \Gamma_{cb} \Gamma_{ba} \ldots \quad \begin{cases} x_d < x_c < x_b < x_a \\ k_{t\,d} \gg k_{t\,c} \gg k_{t\,b} \gg k_{t\,a} \end{cases}$

anti-DGLAP limit $\quad \ldots \Gamma_{cd} \Gamma_{bc} \Gamma_{ab} \ldots \quad \begin{cases} \bar{x}_d > \bar{x}_c > \bar{x}_b > \bar{x}_a \\ k_{t\,d} \ll k_{t\,c} \ll k_{t\,b} \ll k_{t\,a} \end{cases}$

$$= \ldots (\Gamma^T)_{dc} (\Gamma^T)_{cb} (\Gamma^T)_{ba} \ldots .$$

Suggests sym. $K(\gamma, \omega) = K^T(1 + \omega - \gamma, \omega)$. But this $\to$ spurious colour & $1/\omega$ structures, e.g. $\alpha_s^2 C_F^2/\omega^2$ for $g \to q \to g$, in non-ordered limits.

DGLAP attaches $1/\omega$ and colour sum to leg with higher p_t
BFKL attaches them to left-hand leg — **inconsistent**

Matrix requirements

Sensibleness requirement on matrix formulation.

Use similarity transform S to reattach colour and $1/\omega$ factors in anticollinear limit, so as to restore compatibility between DGLAP and BFKL. Resulting symmetry is

$$\mathcal{K}(1+\omega-\gamma,\omega) = S(\omega)\mathcal{K}^T(\gamma,\omega)S^{-1}(\omega)\ .$$

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Other requirements

- K_{qq}, K_{qg} should be free of $1/\omega$ divergences at all orders
- K_{gq}, K_{gg} may at most have $1/\omega$ divergences
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Matrix kernel

Structure quite similar to single-channel; LLx-LO is:

$$\mathcal{K}_0(\gamma,\omega) = \begin{pmatrix} \Gamma_{qq,0}(\omega)\chi_c^\omega(\gamma) & \Gamma_{qg,0}(\omega)\chi_c^\omega(\gamma) + \Delta_{qg}(\omega)\chi_{\mathrm{ht}}^\omega(\gamma) \\ \Gamma_{gq,0}(\omega)\chi_c^\omega(\gamma) & \Gamma_{gg,0}(\omega)\chi_c^\omega(\gamma) + \frac{2C_A}{\omega}\left[\chi_0^\omega(\gamma) - \chi_c^\omega(\gamma)\right] \end{pmatrix}$$

Note $\Delta_{qg}(\omega)$ term: allows one to set *factorisation scheme* at NLO, by modifying the *higher-twist* part of the $\mathcal{K}_{qg}$ kernel.

Without having to add α_s^2/ω term to $\mathcal{K}_{1,qg}$

NB: We choose $\overline{\text{MS}}$

Higher orders:

- Add on $\mathcal{K}_1(\gamma,\omega)$ to get NLx-NLO.
- put in extra higher-twist piece in $\mathcal{K}_0(\gamma,\omega)$ to get α_s^3/ω^2 scheme-dependent terms (NLx-NLO$^+$).

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  Based on study of P_{gg}, CCSS '06

- Formalism 'predicts' that at NLx accuracy, at NNLO

$$\Gamma_{gq,2}^{\mathrm{NL}x} = \frac{C_F}{C_A}\Gamma_{gg,2}^{\mathrm{NL}x}$$

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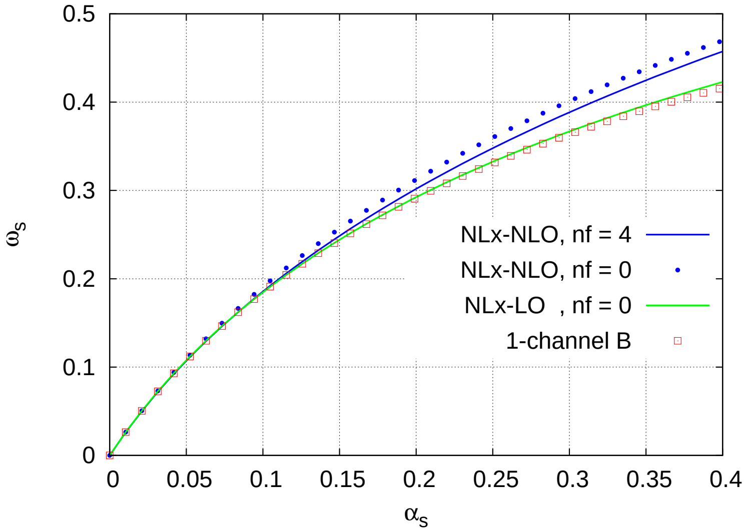
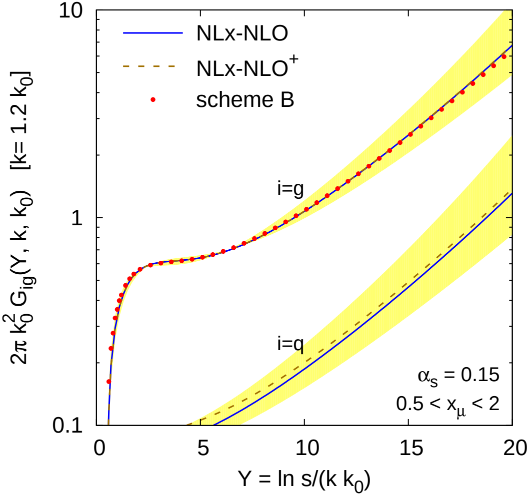
Intercept at fixed coupling

Power of growth of cross-sections and splitting functions at fixed coupling. Rather similar to 2003 results:



Green function



Green function for gluon is very similar to 2003 results.

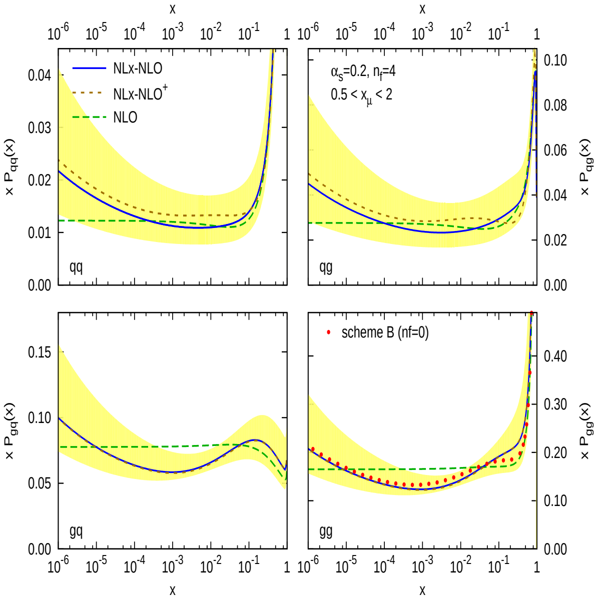
Scale uncertainties (band) under control

Additionally *generate quark component*, with same power-growth, but suppressed by $\sim \alpha_s$.

Scale uncertainties larger — radiative generation

NNLO part of NLx scheme terms (NLO$^+$) have little impact.

Splitting functions



In *gg* channel results again similar to those from 2003

gq channel rather similar to *gg*

Both have **dip** at $x \sim 10^{-3}$

qq and *qg* channels have barely any dip, and large scale uncertainties — NLx is first order of generation of small-x quarks.

Conclusions, outlook

- Have matrix double integral equation that contains both NLx BFKL and NLO DGLAP in $\overline{\text{MS}}$ scheme.
- From it one can deduce Green functions and matrix of effective small-x resummed splitting functions.
- Gluon-channel results agree with earlier resummations, now also get full singlet matrix.

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- providing splitting functions in convenient form for general use
- understanding what happens at NNLO
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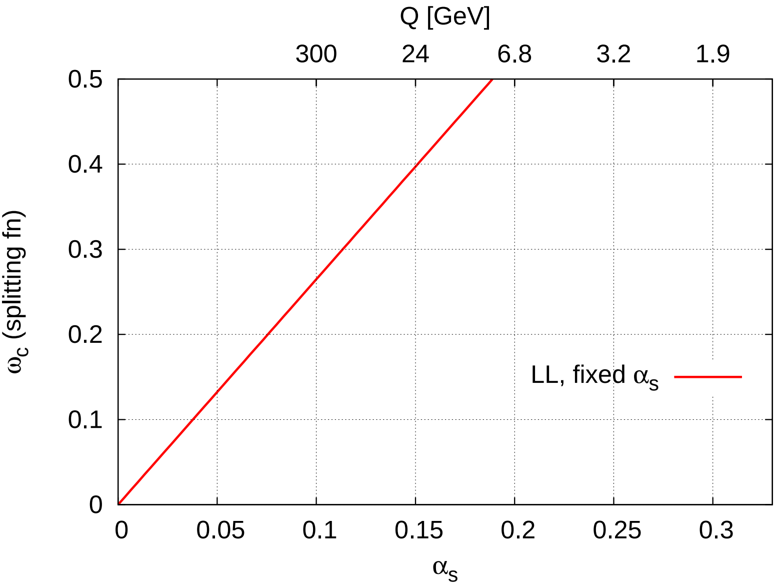
EXTRAS

ω_c vs. α_s



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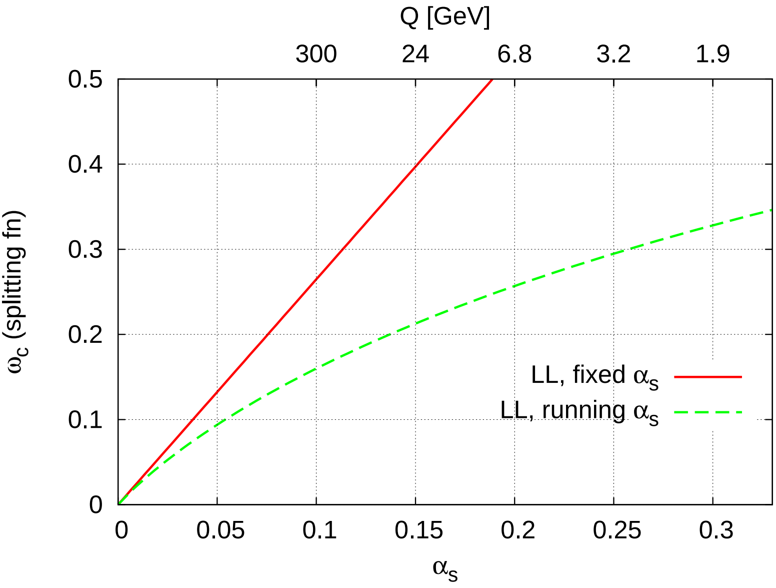

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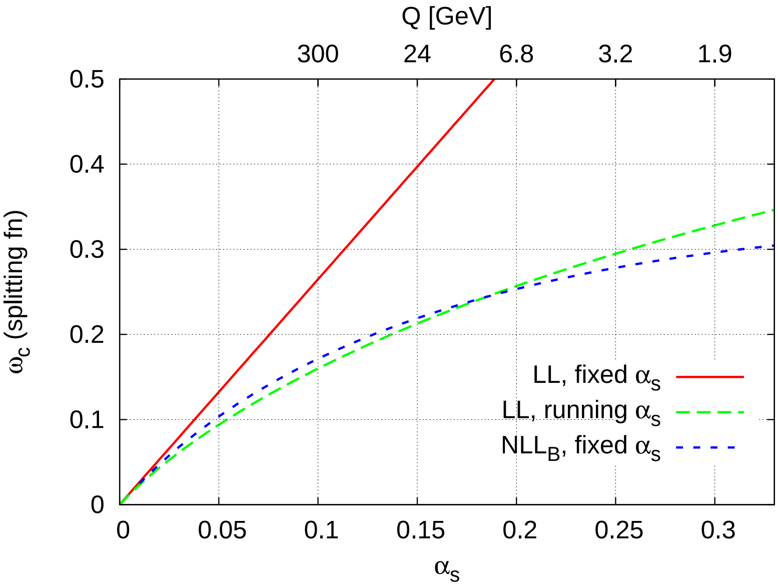

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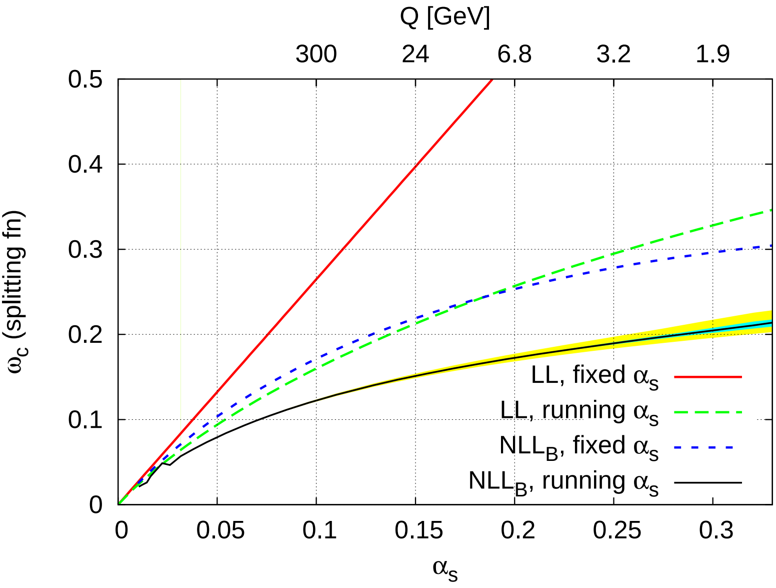

Similarity transforms

$$S = \begin{pmatrix} 2n_f N_c f_q(\omega) & 0 \\ 0 & (N_c^2 - 1) f_g(\omega) \end{pmatrix},$$

$$\overline{\Gamma} = S \Gamma^T S^{-1} = \begin{pmatrix} \Gamma_{qq} & \frac{n_f}{C_F} \frac{f_q(\omega)}{f_g(\omega)} \Gamma_{gq} \\ \frac{C_F}{n_f} \frac{f_g(\omega)}{f_q(\omega)} \Gamma_{qg} & \Gamma_{gg} \end{pmatrix}.$$

$$\mathcal{K} \simeq \frac{\Gamma}{\gamma} + \frac{\overline{\Gamma}}{1 + \omega - \gamma},$$

$$f_q(\omega) = \frac{2T_R}{\omega + 3} \quad \Longrightarrow \quad \overline{\Gamma} = \Gamma,$$

qg entry of matrix kernel

$$\mathcal{K}_{0,qg}(\gamma,\omega) = \Gamma_{qg,0}(\omega)\chi_c^{\omega}(\gamma) + \Delta_{qg}(\omega)\chi_{\rm ht}^{\omega}(\gamma)$$

$\chi_{\rm ht}^{\omega}(\gamma)$ is a higher-twist kernel possessing the $\gamma \leftrightarrow 1+\omega-\gamma$, e.g.

$$\chi_{\rm ht}^{\omega}(\gamma) = \frac{2}{3}\left(\frac{1}{1+\gamma} + \frac{1}{2+\omega-\gamma}\right), \quad \chi_{\rm ht}^{0}(0) = 1\,,$$

Δ_{qg} is an ω-dependent coefficient, regular for $\Re(\omega) > -1$

$$\Delta_{qg}(\omega) \equiv \delta_{qg}\,\Delta(\omega) \equiv \delta_{qg}\cdot 3\left(\frac{1}{1+\omega} - \frac{2}{2+\omega} + \frac{1}{3+\omega}\right), \quad \Delta_{qg}(0) = \delta_{qg}\,.$$

To get the $\overline{\rm MS}$ scheme, set $\delta_{qg} = \delta_{qg}^{\overline{\rm MS}} = 8\,T_f/9$.

Higher-order kernel

$$\mathcal{K}(\alpha_s, \gamma, \omega) \equiv \sum_{n,m,p=0}^{\infty} {}_p\mathcal{K}_n^{(m)} \,\hat{\alpha}^{n+1}\gamma^{m-1}\omega^{p-1}\,, \qquad \hat{\alpha} \equiv \frac{\alpha_s}{2\pi}$$

$$\mathcal{K}_1 = \left(\Gamma_1 - \mathcal{K}_0^{(1)}\mathcal{K}_0^{(0)}\right)\chi_c^{\omega} + (2C_A)^2\left(\frac{1}{\omega} - \frac{2}{1+\omega}\right)\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\chi}_1^{\omega} - \tilde{\chi}_1^{(0)}\chi_c^{\omega} \end{pmatrix}$$

$$\tilde{\chi}_1^{\omega=0} \equiv \tilde{\chi}_1 = \frac{{}_0\mathcal{K}_{gg,1}}{(2C_A)^2} = K_1^{\mathrm{BFKL}} - \frac{\left[{}_0\mathcal{K}_0\;{}_1\mathcal{K}_0\right]_{gg}}{(2C_A)^2}$$

Effective χ — matrix eigenvalues