# Recent developments in small-$x$ physics

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Basic BFKL prediction: perturbative QCD cross sections should grow as $s^{4 \ln 2 \alpha_s N_c / \pi} \sim s^{0.5}$.

Recent history:

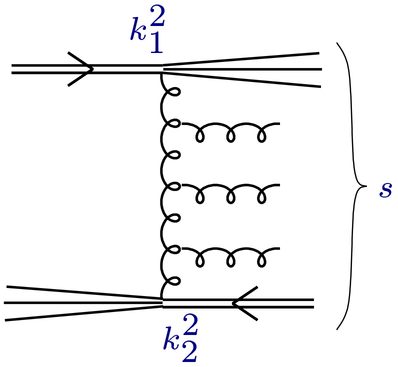
1993–now: Failure to fit a range of data
- ➥ scaling violations
- ➥ forward jets
- ➥ $\gamma^* \gamma^*$ cross section

1989–1998: calculation of NLL corrections
- ➥ oscillating cross sections
- ➥ negative anomalous dimensions

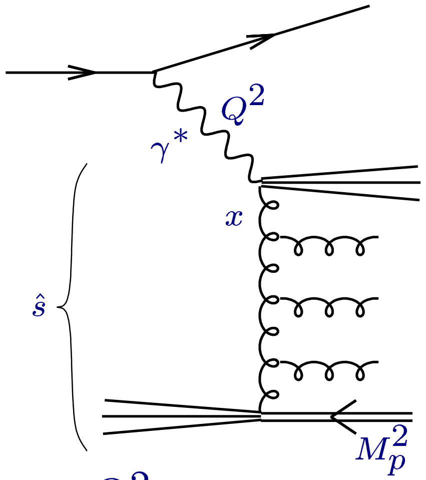
1998–2000: Learning to go beyond NLL
- ➥ much more coherent picture

# High-energy scattering in QCD

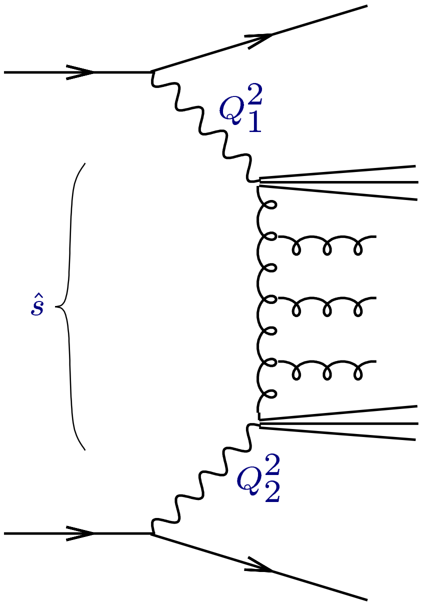
Dominated by gluon exchange in the $t$ channel. Many different contexts




Generic scatttering of two hadronic objects: high energy $\equiv s \gg k_1^2, k_2^2$
Big rapidity interval, $y = \ln \dfrac{s}{k_1 k_2}$.




$x = \dfrac{Q^2}{\hat{s}} \ll 1$, *i.e.* small-$x$ DIS.




High-energy $\gamma^*\gamma^*$ scattering

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Processes dominated by configurations with gluons widely separated in rapidity: rapidity integration for $n$ gluons gives $y^n \sim \ln^n s$ enhancement. We must *resum leading logs (LL)*

$$(\alpha_s \ln s)^n \Rightarrow \sigma \sim s^{4 \ln 2 \bar{\alpha}_s}, \qquad (\bar{\alpha}_s = \alpha_s N_c/\pi \simeq \alpha_s)$$

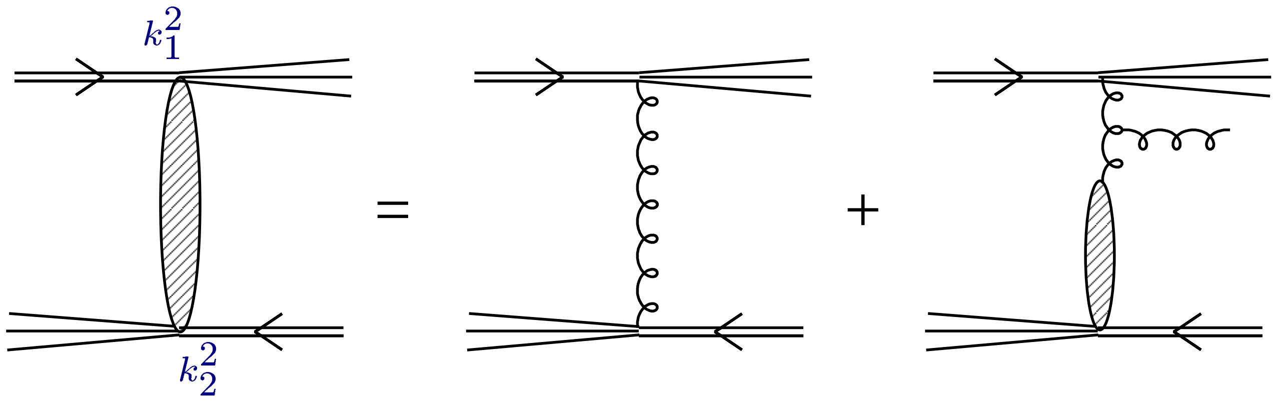
First done in 1976 by Balitsky, Fadin, Kuraev and Lipatov.

## Basic approach

Two aspects to the problem.

- Identifying the basic element of the chain ("the kernel").
- Understanding how it iterates.

Write integral equation for cross section

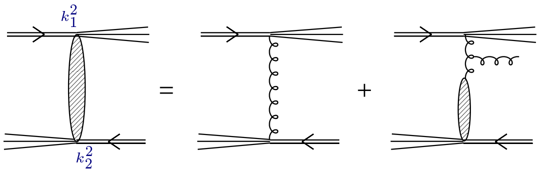



$$\sigma(s, k_1, k_2) \quad = \quad \sigma^{(0)} \quad + \quad \text{Kernel} \otimes \sigma$$

Take Mellin transforms to diagonalise

$$\sigma(s, k_1, k_2) = \int \frac{d\gamma}{2\pi i}\frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega}\frac{1}{k_1^2}\left(\frac{k_1^2}{k_2^2}\right)^{\gamma}\sigma_{\omega,\gamma}$$

- $\ln s$ conjugate to $1/\omega$
- $\ln(k_1^2/k_2^2)$ conjugate to $1/\gamma$
- Define Mellin tranform of Kernel to be $\dfrac{\bar{\alpha}_{\mathrm{s}}\chi(\gamma)}{\omega}$, $\chi(\gamma)$ is the *characteristic function*




$$\sigma_{\omega,\gamma} = \sigma^{(0)}_{\omega,\gamma} + \frac{\bar{\alpha}_{\mathrm{s}}}{\omega}\chi(\gamma)\cdot\sigma_{\omega,\gamma} \quad \Rightarrow \quad \sigma_{\omega,\gamma} = \frac{\omega\sigma^{(0)}_{\omega,\gamma}}{\omega - \bar{\alpha}_{\mathrm{s}}\chi(\gamma)}$$

Close $\omega$ contour and we get

$$\boxed{\omega = \bar{\alpha}_{\mathrm{s}}\chi(\gamma)}$$

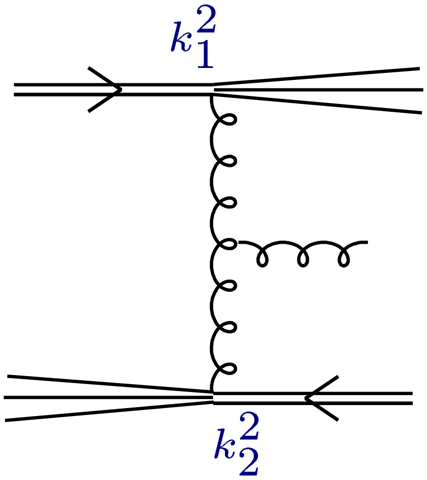
Cross section goes as

$$\sigma(s,k_1^2,k_2^2) \sim \int \frac{d\gamma}{2\pi i}\left(\frac{s}{s_0}\right)^{\bar{\alpha}_{\mathrm{s}}\chi(\gamma)} \frac{1}{k_1^2}\left(\frac{k_1^2}{k_2^2}\right)^{\gamma}\sigma^{(0)}_{\gamma}$$

It must be symmetric under $k_1 \leftrightarrow k_2$

$$\frac{1}{k_1^2}\left(\frac{k_1^2}{k_2^2}\right)^{\gamma} = \frac{1}{k_2^2}\left(\frac{k_2^2}{k_1^2}\right)^{1-\gamma} \Rightarrow \boxed{\chi \text{ is symmetric } \gamma \leftrightarrow 1-\gamma}$$

# Characteristic function




*Study a single branching.* Two cases:

$k_1 \gg k_2$
$k_1 \ll k_2$

When $k_1 \gg k_2$, cross section goes as

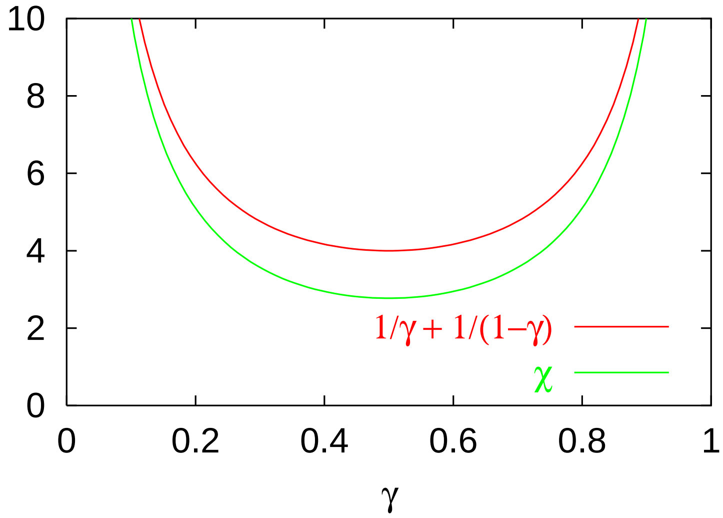
$$\sigma \sim \frac{\bar{\alpha}_s}{k_1^2} \ln\frac{s}{s_0} \ln\frac{k_1^2}{k_2^2} \qquad \Rightarrow \qquad \chi \simeq \frac{1}{\gamma}$$

When $k_2 \gg k_1$, cross section goes as

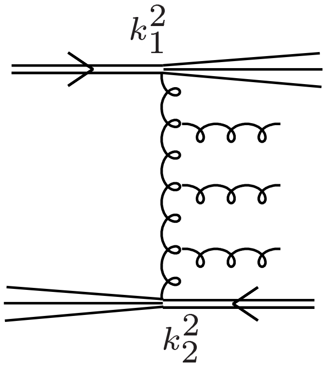
$$\sigma \sim \frac{\bar{\alpha}_s}{k_2^2} \ln\frac{s}{s_0} \ln\frac{k_2^2}{k_1^2} \qquad \Rightarrow \qquad \chi \simeq \frac{1}{1-\gamma}$$

Full $\chi$ is

$$\chi = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

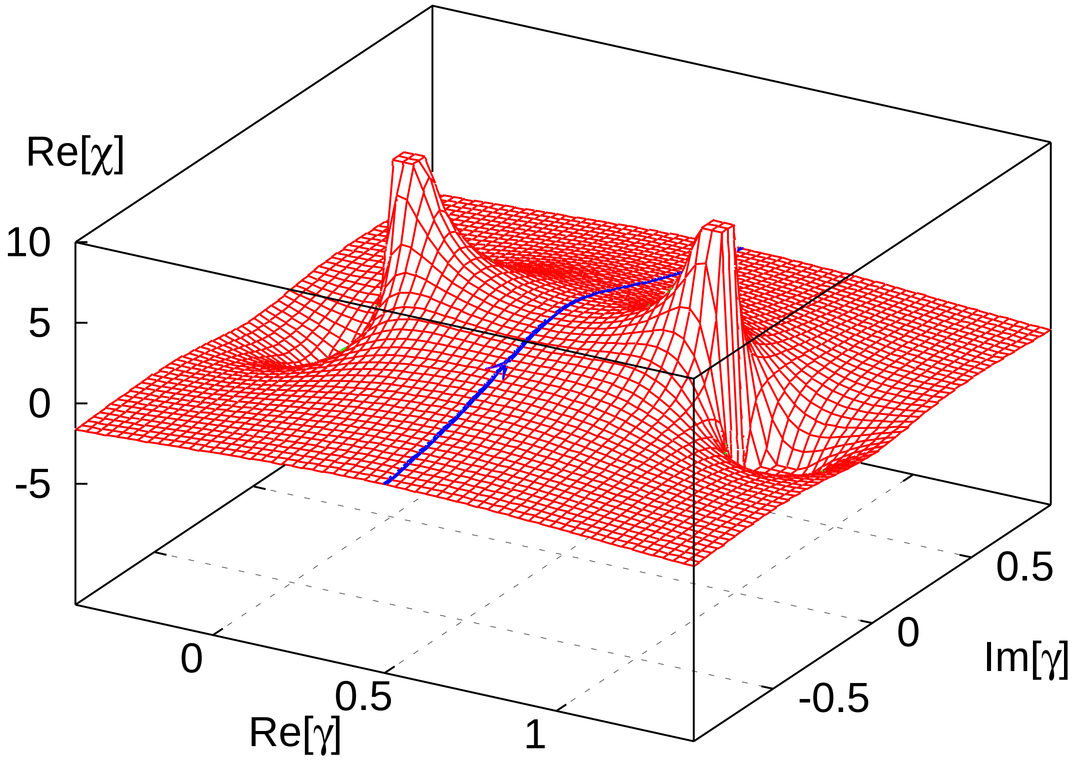


# Inverse Mellin transform




$$\frac{\omega}{\bar{\alpha}_s} = \chi = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$\sigma(s, k_1^2, k_2^2) \sim \int_C \frac{d\gamma}{2\pi i} \left(\frac{s}{s_0}\right)^{\bar{\alpha}_s \chi(\gamma)} \frac{1}{k_1^2} \left(\frac{k_1^2}{k_2^2}\right)^{\gamma}$$

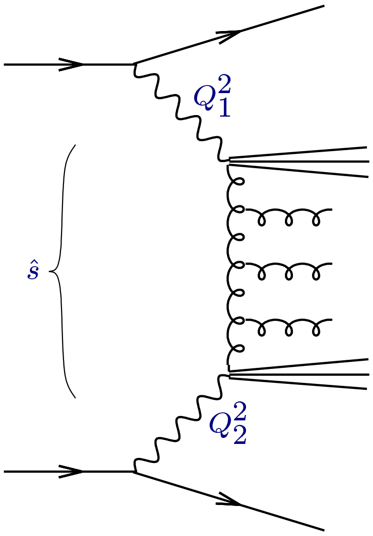



- For large $s$, integral is dominated by saddle-point at $\gamma = \frac{1}{2}$, where $\chi(\frac{1}{2}) = 4 \ln 2$.

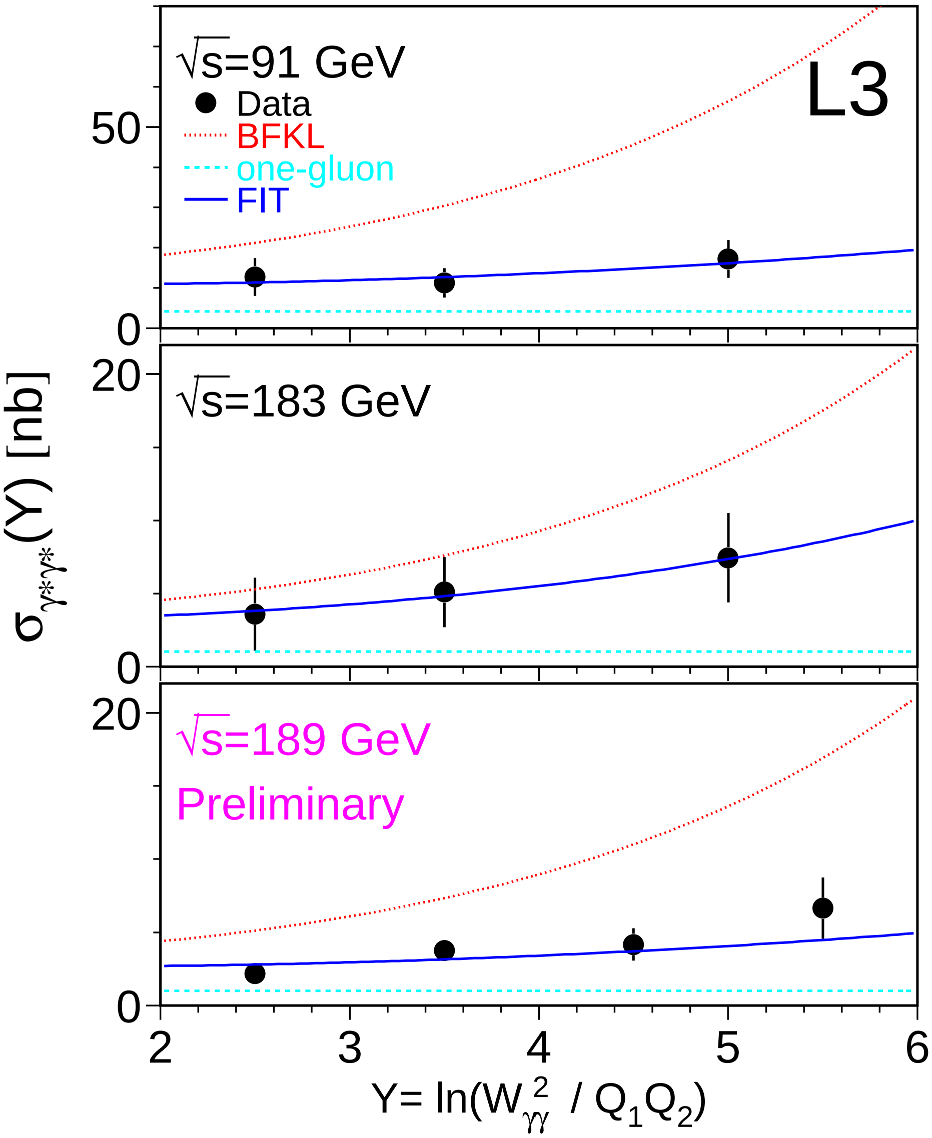
- So cross section has form $\sigma \sim \dfrac{1}{k_1 k_2} \left(\dfrac{s}{s_0}\right)^{\bar{\alpha}_s \chi(\frac{1}{2})}$

# Phenomenological implications

Cross sections should grow as $s^{\alpha_s 4 \ln 2} \simeq s^{0.5}$ for $\alpha_s \simeq 0.2$.



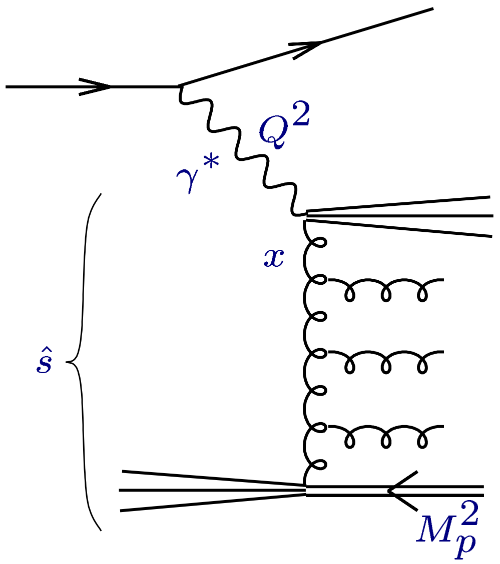

$\gamma^*\gamma^*$ scattering measured by L3 collaboration at $\langle Q_1 Q_2 \rangle \simeq 3.5,\ 14$ and $14.5\,\mathrm{GeV}^2$.



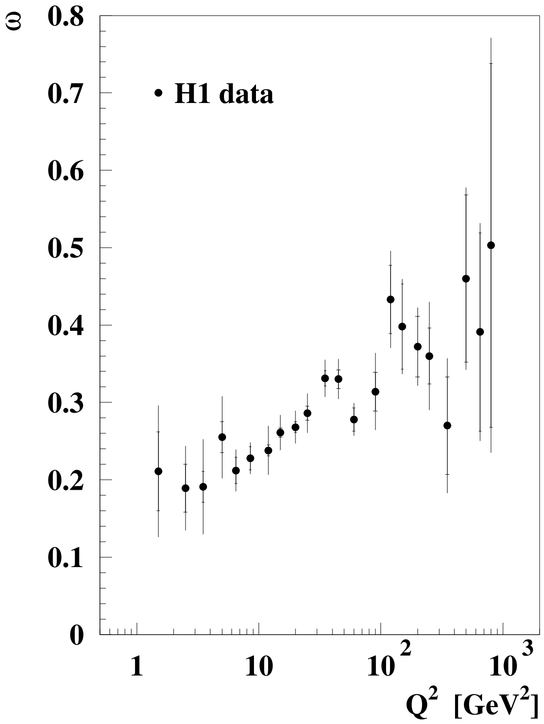

Single-gluon exchange is insufficient

BUT BFKL RISES TOO FAST!

# DIS structure functions




Presence of proton means problem is non-perturbative (NP). Formally cannot use BFKL. But upper part of evolution is perturbative (PT), so we might still expect to see BFKL power growth.




Small-$x$ power growth ($\omega$) of $F_2$ has been measured by HERA: well below $0.5$. [For large $Q^2$ it increases: due to double logs, $(\alpha_s \ln x \ln Q^2)^n$.]

Do things properly: only scaling violations are predictable, so study anomalous dimensions, $\gamma(\omega)$. BFKL resummation predicts the terms $(\alpha_s/\omega)^n$, through the formula:

$$\gamma(\omega) = \chi^{-1}\left(\frac{\omega}{\bar{\alpha}_s}\right) = \frac{\bar{\alpha}_s}{\omega} + 2\zeta(3)\left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + \cdots$$

# DIS scaling violations

$\gamma(\omega) = \chi^{-1}\left(\frac{\omega}{\bar{\alpha}_s}\right)$ has branch point at $\omega = 4\ln 2\bar{\alpha}_s \Rightarrow$ evolution in $Q^2$ should eventually cause structure functions to grow as $x^{-4\ln 2\bar{\alpha}_s}$.

May not have seen it in previous plot because combinations of $x$ and $Q^2$ not sufficiently asymptotic. So explicitly check the scaling violations:

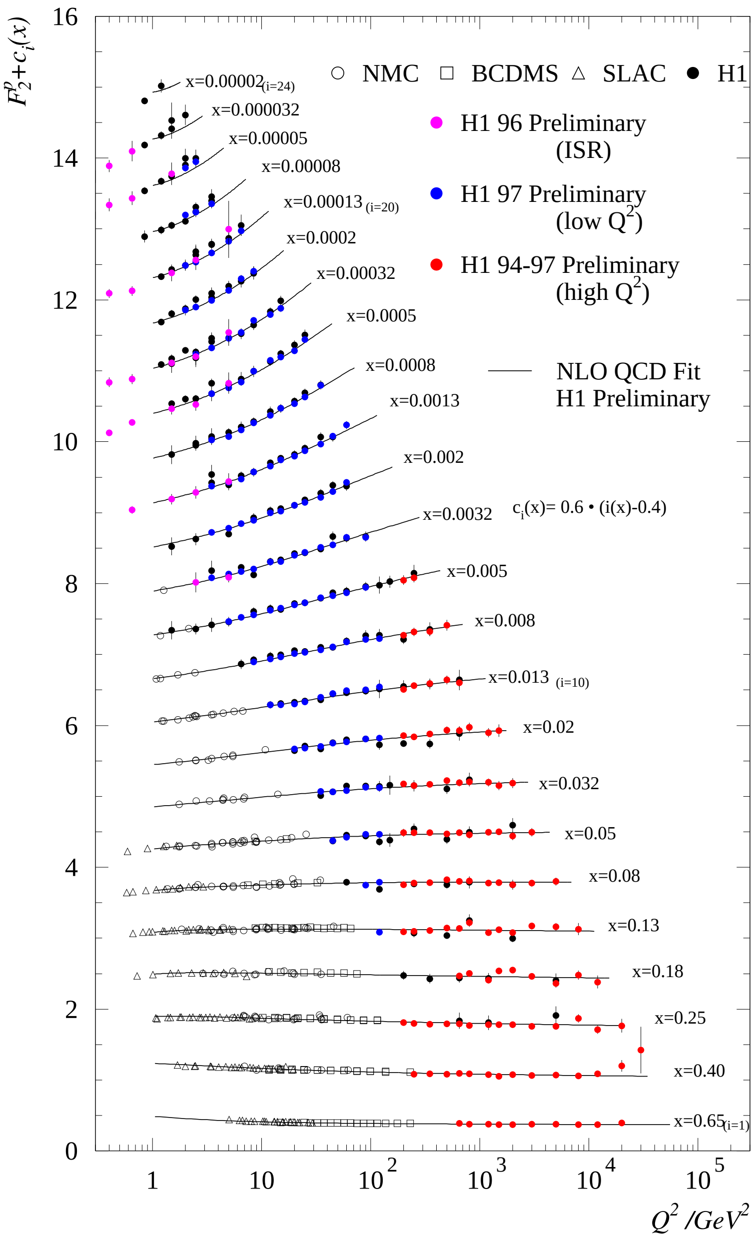



Figure shows fit with just DGLAP and no small-$x$ resummation.

It works perfectly!

Ball & Forte (1996) tried fits including BFKL resummation: predicted scaling violations are too large.

Conclusions: BFKL always overestimates X-section growth.

*WHY?*

# Next-to-leading corrections

Possible reason is that BFKL evolution has siginificant higher-order corrections ⇒ determine next-to-leading log (NLL) corrections

$$\boxed{\alpha_s(\alpha_s \ln s)^n}$$

Origins: anything which gives a power of $\alpha_s$ without an extra integration over rapidity, *e.g.*

two particles emitted close in rapidity
running coupling effects
...

Actual calculation involves many separate pieces

2-loop correction to gluon regge trajectory
1-loop correction to RRG vertex
RRGG vertex
RRQQ vertex

R=reggeised gluon ("$t$-channel gluon together with its virtual corrections"), P=parton, G=gluon, Q=quark.

Final result is $\chi_1(\gamma)$, the $\mathcal{O}(\alpha_s)$ correction to the characteristic function:

$$\chi(\gamma) = \chi_0(\gamma) + \bar{\alpha}_s \chi_1(\gamma)$$

# 10 years' work!

L.N. Lipatov and V.S. Fadin, *Sov. J. Nucl. Phys.* **50** (1989) 712; V.S. Fadin, R. Fiore and M.I. Kotsky, *Phys. Lett.* **B 539** (1995) 181; V.S. Fadin, R. Fiore and M.I. Kotsky, *Phys. Lett.* **B 387** (1996) 593 [hep-ph/9605357]; V.S. Fadin, and L.N. Lipatov, *Nucl. Phys.* **B 406** (1993) 259; V.S. Fadin, R. Fiore and A. Quartarolo, *Phys. Rev.* **D 50** (1994) 5893 [hep-th/9405127]; V.S. Fadin, R. Fiore, and M.I. Kotsky, *Phys. Lett.* **B 389** (1996) 737 [hep-ph/9608229]; V.S. Fadin and L.N. Lipatov, *Nucl. Phys.* **B 477** (1996) 767 [hep-ph/9602287]; V.S. Fadin, M.I. Kotsky and L.N. Lipatov, *Phys. Lett.* **B 415** (1997) 97; I.A. Korchemskaya and G.P. Korchemsky, *Phys. Lett.* **B 387** (1996) 346 [hep-ph/9607229]; J. Blumlein, V. Ravindran and W.L. van Neerven, *Phys. Rev.* **D 58** (1998) 091502 [hep-ph/9806357]; S. Catani, M. Ciafaloni and F.Hautman, *Phys. Lett.* **B 242** (1990) 97; S. Catani, M. Ciafaloni and F.Hautman, *Nucl. Phys.* **B 366** (1991) 135; V.S. Fadin, R. Fiore, A. Flashi, and M.I. Kotsky, *Phys. Lett.* **B 422** (1998) 287 [hep-ph/9711427]; V. Del Duca, *Phys. Rev.* **D 54** (1996) 989; V. Del Duca, *Phys. Rev.* **D 54** (1996) 4474; V. Del Duca and C.R. Schmidt, *Phys. Rev.* **D 57** (1998) 4069 [hep-ph/9711309]; V. Del Duca and C.R. Schmidt, hep-ph/9810215.

Put together in 1998:

M. Ciafaloni and G. Camici, *Phys. Lett.* **B 430** (1998) 127 [hep-ph/9803389];
M. Ciafaloni, *Phys. Lett.* **B 429** (1998) 349 [hep-ph/9801322];
G. Camici and M. Ciafaloni, *Phys. Lett.* **B 412** (1997) 396 [hep-ph/9707390];
V.S. Fadin and L.N. Lipatov, *Phys. Lett.* **B 429** (1998) 127 [hep-ph/9802290].

# We will guess 'most' of it in 10 minutes ...

More precisely, find the collinearly and anti-collinearly enhanced pieces, *i.e.* the structure of singularities around $\gamma = 0$ and $\gamma = 1$.

# NLL — running $\alpha_s$

At NLO, $\bar{\alpha}_s({k_2}^2) = \bar{\alpha}_s(k_1^2) - b\bar{\alpha}_s^2 \ln \frac{k_2^2}{k_1^2}$

($b = 11/12$ is first coefficient of $\beta$-function)

Correct renormalisation scale is largest of $k_1$ and $k_2$

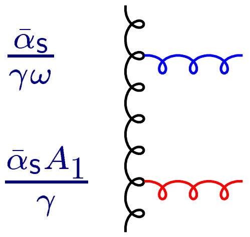
$$k_1 \gg k_2 \;\Rightarrow\; \frac{\bar{\alpha}_s(k_1^2)}{\omega} \ln \frac{k_1^2}{k_2^2} \;\Rightarrow\; \frac{\bar{\alpha}_s(k_1^2)}{\omega} \frac{1}{\gamma}$$

$$k_1 \ll k_2 \;\Rightarrow\; \frac{\bar{\alpha}_s(k_2^2)}{\omega} \ln \frac{k_2^2}{k_1^2} \;\Rightarrow\; \frac{\bar{\alpha}_s(k_1^2)}{\omega} \left( \frac{1}{1-\gamma} - \frac{\bar{\alpha}_s b}{(1-\gamma)^2} \right)$$

# DGLAP splitting function

Mellin transform of DGLAP splitting function is:

$$\gamma^{DGLAP}(\omega) = \bar{\alpha}_s \left( \frac{1}{\omega} + A_1 + \mathcal{O}(\omega) \right), \qquad (A_1 = -11/12)$$




Combine small-$x$ ($\bar{\alpha}_s/\omega\gamma$) and finite-$x$ branchings ($\bar{\alpha}_s A_1/\gamma$) to get NLO contribution

So NLL collinearly-enhanced splitting-function part is

$$\frac{\bar{\alpha}_s^2 A_1}{\omega} \left( \frac{1}{\gamma^2} + \frac{1}{(1-\gamma)^2} \right)$$

# NLL — choice of energy scale $s_0$

We resum terms $(\bar\alpha_s \ln s/s_0)^n$ — modifying $s_0$,$s_0 \to s_0'$ is equivalent to making a subleading correction, *e.g.*

$$\frac{1}{4}\left(\bar\alpha_s \ln \frac{s}{k_1 k_2} \ln \frac{k_1^2}{k_2^2}\right)^2$$

$$\to \frac{1}{4}\left(\bar\alpha_s \ln \frac{s}{k_1^2} \ln \frac{k_1^2}{k_2^2}\right)^2 + \frac{\bar\alpha_s^2}{4} \ln \frac{s}{k_1^2} \ln^3 \frac{k_1^2}{k_2^2} + \mathcal{O}\left(\alpha_s^2\right)$$

*BUT:* strong constraint from renormalisation group. If $k_1^2 \gg k_2^2$, we know that cross section contains only terms

$$\sigma \sim \frac{1}{k_1^2} \sum C_n \left(x^{-1} = \frac{s}{k_1^2}\right) \left(\alpha_s \ln \frac{k_1}{k_2}\right)^n + \mathcal{O}\left(\frac{k_2^2}{k_1^4}\right).$$

Terms $\alpha_s^2 \ln^3 k_1/k_2$ (double collinear logs) *are not allowed*.

We must subtract them from expression with $s_0 = k_1 k_2$, giving NLL correction:

$$-\frac{\bar\alpha_s^2}{2\omega}\left(\frac{1}{\gamma^3} + \frac{1}{(1-\gamma)^3}\right)$$

(exact coefficient actually requires an extra 5 minutes...)

# The *REAL THING*

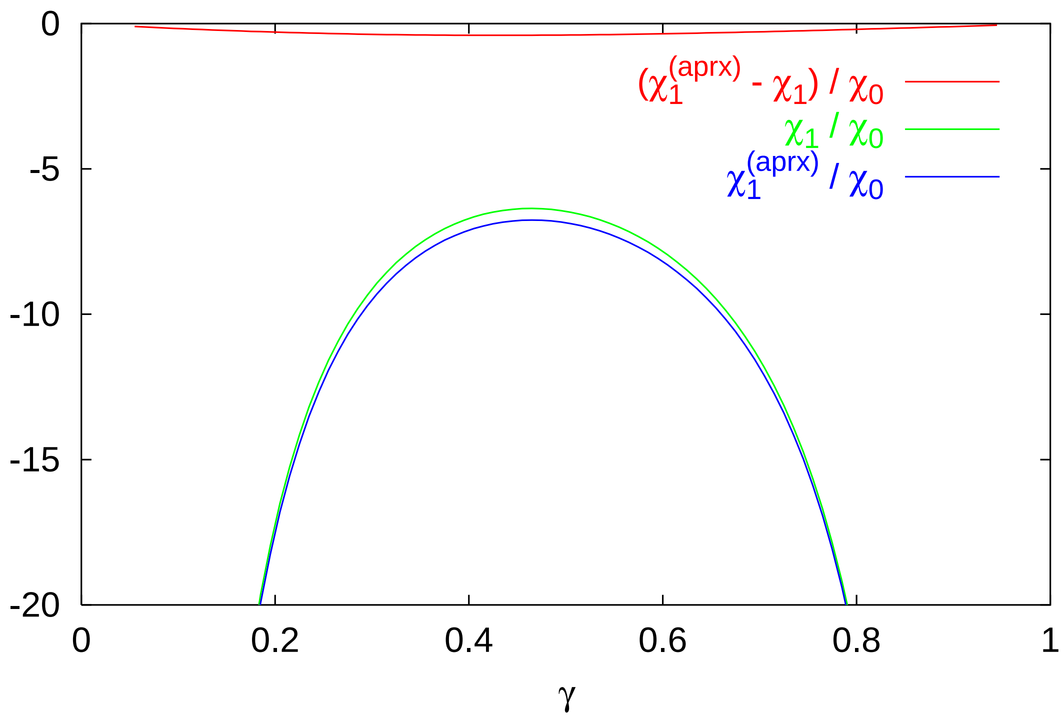
$$\chi_1(\gamma) = -\frac{\pi^2 \cos(\pi\gamma)}{4\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$
$$-\frac{b}{2}\left(\chi_0^2(\gamma) - \psi'(\gamma) + \psi'(1-\gamma)\right) + \frac{\psi''(\gamma)}{4} + \frac{\psi''(1-\gamma)}{4}$$
$$+\left(\frac{67}{36} - \frac{\pi^2}{12}\right)\chi_0(\gamma) + \frac{3}{2}\zeta(3) + \frac{\pi^3}{4\sin(\pi\gamma)} - \phi(\gamma)$$

where

$$\phi(\gamma) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2} + \frac{\psi(n+2-\gamma)-\psi(1)}{(n+1-\gamma)^2}\right]$$

Our approximation was

$$\chi_1^{(\mathrm{aprx})} = \frac{A_1}{\gamma^2} + \frac{A_1}{(1-\gamma)^2} - \frac{b}{(1-\gamma)^2} - \frac{1}{2\gamma^3} - \frac{1}{2(1-\gamma)^3}$$

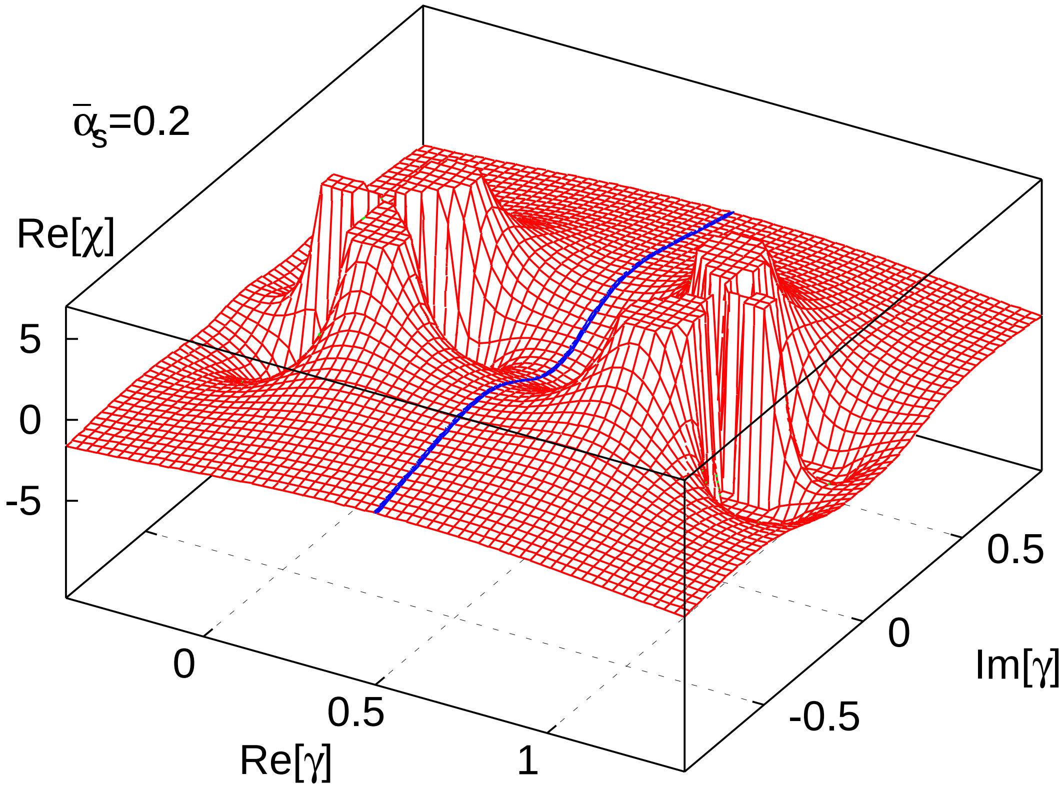


# Consequences of NLL

$$\bar{\alpha}_s \chi(\tfrac{1}{2}) \simeq \bar{\alpha}_s \chi^{LL}(\tfrac{1}{2})(1 - 6.47\bar{\alpha}_s) \qquad (n_f = 0)$$

for $\alpha_s = 0.2$ we have $\omega_c^{NLL} \simeq -0.125$

Look at structure of $\chi$ and inverse Mellin transform:




Integral has two saddle-points at *complex* $\gamma = \frac{1}{2} \pm i\nu_0$.

D.A. Ross, 1998

$$\sigma(s, k_1^2, k_2^2) \sim \frac{1}{k_1^2}\left(\frac{k_1^2}{k_2^2}\right)^{1/2 \pm i\nu_0} \sim \frac{1}{k_1 k_2} \cos\left(\nu_0 \ln \frac{k_1^2}{k_2^2}\right)$$

Happens when $\chi''(\frac{1}{2}) < 0$: $\chi''(\frac{1}{2}) \simeq \chi''^{LL}(\frac{1}{2})(1 - 19.2\bar{\alpha}_s)$

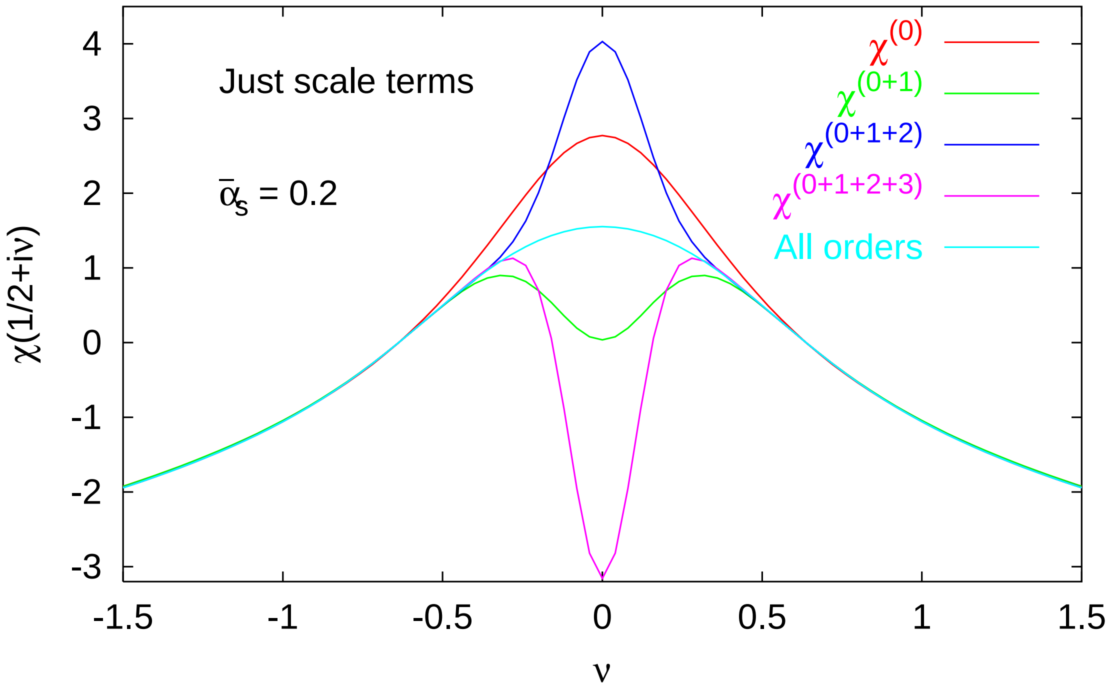
# What's going on?

➥ Slow convergence of the PT series?

It can't be anything else

➥ Should we calculate NNLL?

LL $\simeq$ 1 year
NLL $\simeq$ 10 years
$19 \lesssim$ NNLL $\lesssim 100$ years
Besides it might not even help...




expansion of LDC Model (Andersson, Gustafson & Samuelson, 1996)

➥ Would like extra resummation — but of what?

Various proposals include BLM scale setting with 'appropriate scheme' (Brodsky, Fadin, Kim, Lipatov & Pivovarov, 1998) and enforcing a rapidity gap between emissions (Schmidt, 1999).

# Collinear resummation

Worst problems arise in (anti)-collinear regions, $k_1 \gg k_2$ and $k_1 \ll k_2$. ($\gamma = 0$ and $\gamma = 1$ in Mellin transform space).

If we want to describe those regions *we must resum the collinearly-enhanced terms*.

But we also know that the first couple of collinearly enhanced terms are very successful at reproducing the full NLL corrections even at $\gamma = 1/2$ ($> 93\%$ of $\chi_1$; $99.5\%$ of $\chi_1''$!). So a collinear resummation might solve problems even there.

The terms that we want to resum are:

- Scale terms $\alpha_s^n/\gamma^{2n+1}$

GPS 1998

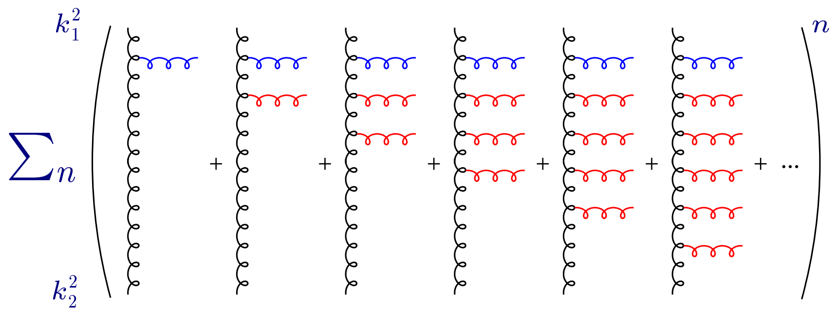
- Splitting function $\alpha_s^n/\gamma^{n+1}$
- Running coupling $\alpha_s^n/\gamma^{n+1}$

Ciafaloni and Colferai, 1998
Ciafaloni, Colferai and GPS, 1999
Altarelli, Ball and Forte, 1999

plus corresponding terms divergent at $\gamma = 1$ (related through symmetry).

# Splitting function and running $\alpha_s$ terms

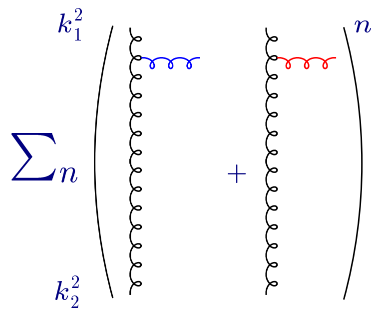
Look at collinear limit $k_1^2 \gg k_2^2$:




BLUE emission is small-$x$: $\frac{\alpha_s(k_1^2)}{\omega\gamma}$

RED emissions are finite-$x$: $\frac{\alpha_s(k^2)A_1}{\gamma}$

$\Rightarrow$




$+$ corrections from ends of chain, free of $\alpha_s/\omega$ factor.

Ciafaloni & Colferai 1998

Ciafaloni, Colferai & GPS, 1999

Go from $\alpha_s$ expansion to $\omega$ expansion :

$$\left(\frac{\bar{\alpha}_s}{\omega\gamma} + \frac{\bar{\alpha}_s^2 A_1}{\omega\gamma^2} + \cdots\right) \Rightarrow \bar{\alpha}_s\left(\frac{1}{\omega\gamma} + \frac{A_1}{\gamma}\right) = \frac{\bar{\alpha}_s}{\omega\gamma}(1 + A_1\omega)$$

Essentially equivalent to replacing $1/\omega$ with full collinear splitting function (actually use $A_1 = A_1(\omega)$).

Also takes care of higher order running coupling corrections.

Similar procedure for anti-collinear region

$$\left(\frac{\bar{\alpha}_s}{\omega(1-\gamma)} + \frac{\bar{\alpha}_s^2(A_1 - b)}{\omega(1-\gamma)^2} + \cdots\right) \Rightarrow \frac{\bar{\alpha}_s}{\omega(1-\gamma)}(1 + (A_1 - b)\omega)$$

# Energy scale terms

Changes of energy scale closely related to shifts in $\gamma$:

$$\left(\frac{s}{k_1 k_2}\right)^{\omega}\left(\frac{k_1^2}{k_2^2}\right)^{\gamma} = \left(\frac{s}{k_1^2}\right)^{\omega}\left(\frac{k_1^2}{k_2^2}\right)^{\gamma-\omega/2}$$

It turns out that the following kernel, with shifted poles, is free of unwanted double logs in the collinear and anti-collinear limits (defined for $s_0 = k_1 k_2$:

$$\chi_0^{\omega}(\gamma) = 2\psi(1) - \psi(\gamma+\omega/2) - \psi(1-\gamma+\omega/2)$$

first discussed by Andersson, Gustafson & Samuelsson, 1996

Check collinear limit: $\dfrac{1}{\gamma+\omega/2} \Rightarrow \dfrac{1}{\gamma} - \dfrac{\bar{\alpha}_s}{2\gamma^3}$

General procedure is to shift all poles as follows:

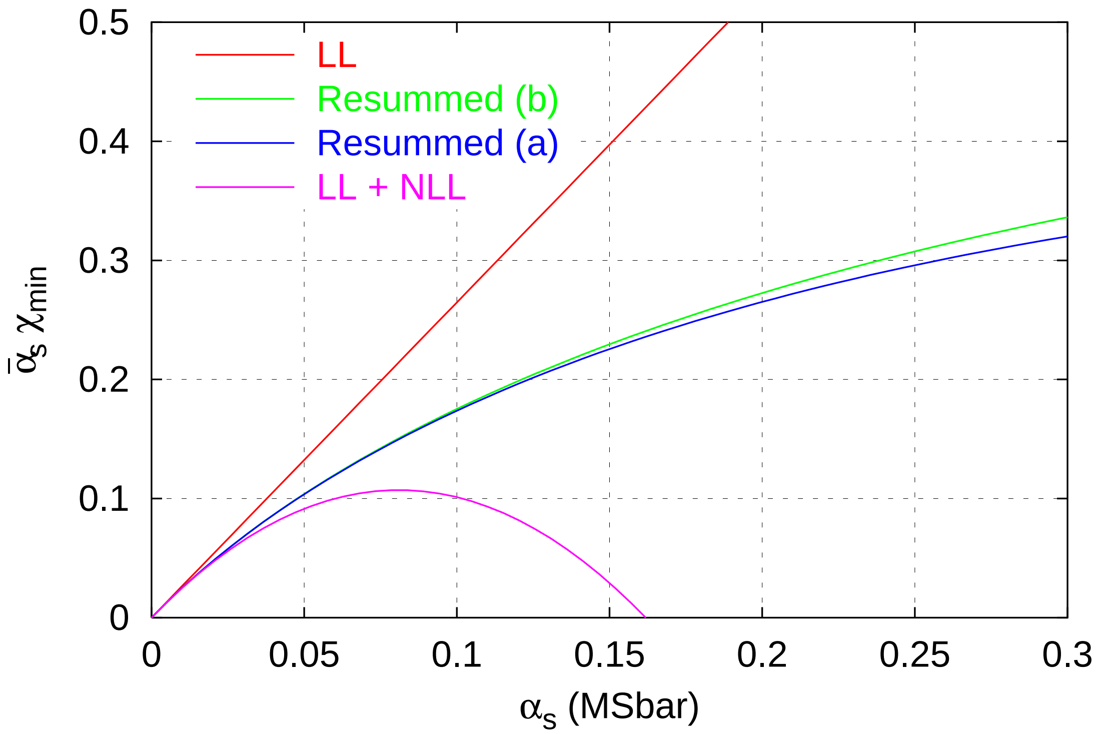
$$\frac{1}{\gamma^n} \to \frac{1}{(\gamma+\omega/2)^n}, \qquad \frac{1}{(1-\gamma)^n} \to \frac{1}{(1-\gamma+\omega/2)^n}$$

This *guarantees* a result free of forbidden double collinear logs.

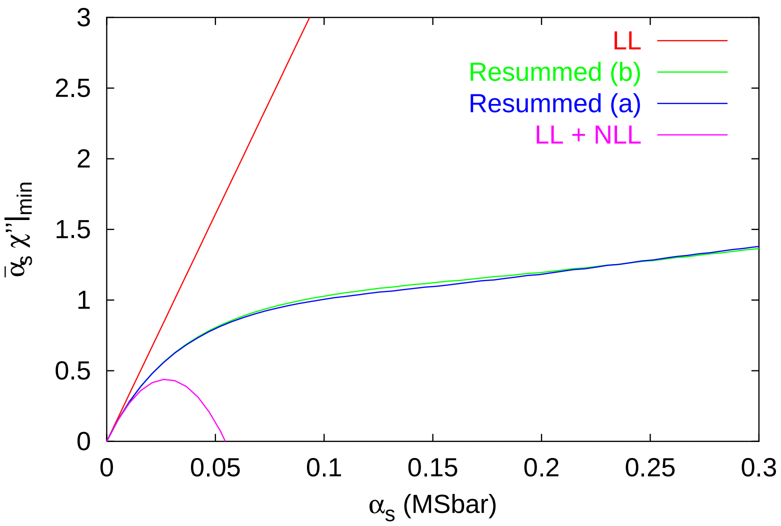
GPS 1999

# Properties of resummed $\chi$

Resummation does not fully specify higher-order terms. There is a remaining uncertainty from non-collinearly enhanced terms. Conventions about exactly what terms to include leads to slightly different answers (resummations a and b).




Lower power is very much in agreement with what's seen experimentally (*e.g.* L3 data — $\alpha_s(Q^2) \simeq 0.25$ — has $\omega \simeq 0.30$).

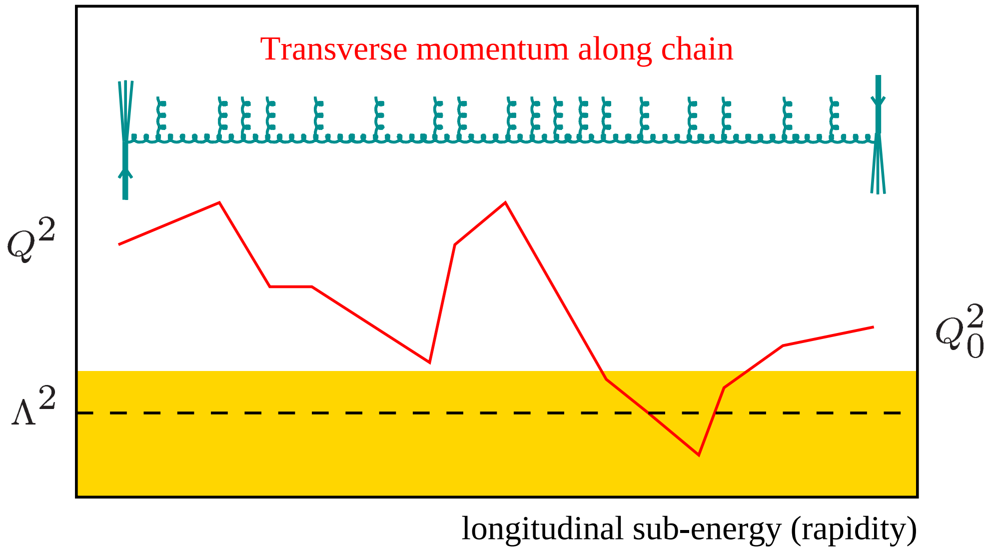



Second derivative now stable: *i.e.* X-sections don't oscillate

# Only half the story!

We've understood the kernel: other part of problem is to figure out how it iterates.

Running $\alpha_s$ complicates the problem enormously. Why?



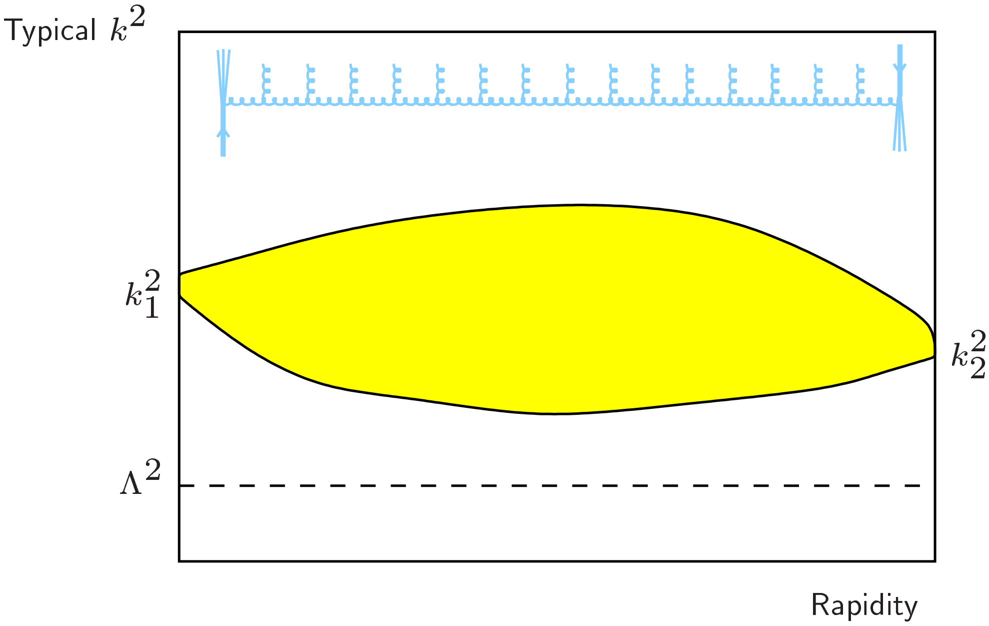

Evolution samples a range of scales (*diffusion*).

Kernel depends on $\alpha_s(k^2)$ (up to now we have relied on its scale invariance): different evolution at different scales.

At high energies non-perturbative (NP) region dominates the evolution ($\alpha_s$ is largest $\Rightarrow$ power growth is largest).

# Diffusion and running $\alpha_s$

Useful representation for discussion is 'Bartels cigar'. Contour plot of transverse momenta sampled in evolution




To a first approximation $\ln k^2$ undergoes a random walk $\Rightarrow$ diffusion. Width (in $\ln k^2$) of cigar is $\sim \sqrt{\alpha_s D Y}$

What are consequences for evolution with running $\alpha_s$?

Upper part of cigar grows as $\exp[\bar{\alpha}_s \chi_{\min}(1 - b\bar{\alpha}_s\sqrt{\alpha_s D Y})Y]$
Lower part of cigar grows as $\exp[\bar{\alpha}_s \chi_{\min}(1 + b\bar{\alpha}_s\sqrt{\alpha_s D Y})Y]$

To first order difference cancels. To second order we get a contribution:

$$\bar{\alpha}_s^5 Y^3 \chi_{\min}^2 b^2 D$$

➥Cross section does not grow as simple power, but rather

$$\sigma \sim \exp[\bar{\alpha}_s(k_1 k_2)\chi_{\min} Y + \bar{\alpha}_s^5 Y^3 b^2 \chi_{\min}^2 D]$$

Kovchegov & Mueller, 1998; Levin, 1998
Ciafaloni, Colferai & GPS, 1999

# Diffusion is not all

Diffusion holds in approximation of *small steps*.

But steps in $\ln k^2$ can also be large (collinear branching).

Question: at what rapidity does NP start to play a role.

Diffusion is very slow at reaching NP: happens when power growth from previous page is of order $1$, *i.e.* $\alpha_s^5 Y^2 \sim 1$, or $Y \sim \alpha_s^{5/2}$.

But evolution can *'tunnel'* to NP region: costs $\Lambda^2/k^2$.

Evolution at low (NP) scales has a power growth $\omega_{\mathbb{P}} \gg \bar{\alpha}_s(k^2)\chi_{\min}$. At a certain rapidity this compensates the cost of tunneling, and NP physics dominates the cross section:

$$\boxed{Y_{\text{NP}} \simeq \frac{\ln k^2/\Lambda^2}{\omega_{\mathbb{P}} - \bar{\alpha}_s(k^2)}}$$

Illustrated with very extreme kinematics in the figures for the evolution of the 'cigar'.

# Anomalous dimensions

DIS cross sections are not PT: depend on hadron structure.

But we have OPE/factorisation: predict $Q^2$-evolution of hadron structure except for higher twist terms of relative order $\lambda^2/Q^2$. Why? dominant contribution is a chain ordered in $k^2$. Unordered chains are suppressed.

At small $x$, unordered chains are the norm. So what happens to factorisation? *It could well break down.*

Mueller 1996

Can use a number of techniques to study the problem:

- Lipatov's (1986) running-coupling eigenfunctions.
- Toy models: solvable.
- Numerically solve BFKL to get $g(x,Q^2)$ and $\partial_{Q^2} g(x,Q^2)$ $\Rightarrow$ deconvolute to get $P(x,Q^2)$.

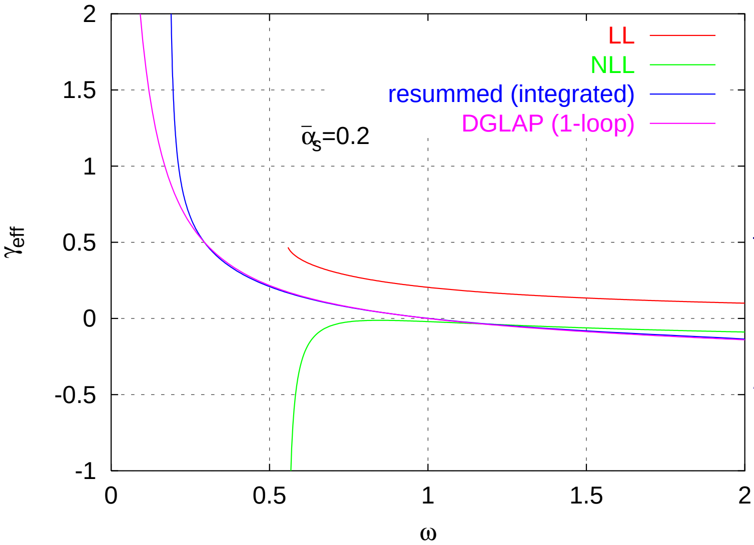
Ciafaloni, Colferai & GPS, 1999

Ciafaloni, Colferai, GPS & Taiuti, 2000

A coherent picture is coming out:

- Factorisation does hold. Corrections to predicted scaling violations are relative $1/Q^2$ (without damaging small-$x$ enhancements).
- Anomalous dimensions are very different from those with fixed coupling.

The small-$x$ anomalous dimension goes as




Momentum sum rule holds

$$P_{gg}(x) \sim \frac{\bar{\alpha}_s}{x} + \mathcal{O}\left(\frac{\bar{\alpha}_s^2}{x^{1+\omega_c}}\right)$$

$$\gamma_{gg}(\omega) \sim \frac{\bar{\alpha}_s}{\omega} + \mathcal{O}\left(\frac{\bar{\alpha}_s^2}{\omega - \omega_c}\right)$$

Need to go to very small $x$ before second term dominates.

Additionally $\omega_c(Q^2)$ is significantly smaller than normal BFKL power $\bar{\alpha}_s(Q^2)\chi_{\min}$:

$$\omega_c(Q^2) = \bar{\alpha}_s(Q^2)\chi_{\min} - 11.2\bar{\alpha}_s^{5/3}.$$

Running coupling corrections are larger than NLL corrections!

What is going on? Evolution below $Q^2$ is factorised out into initial condition. Only evolution above $Q^2$ is left $\Rightarrow$ effective scale for $\alpha_s$ is much larger $\Rightarrow$ power growth of anomalous dimension is suppressed.

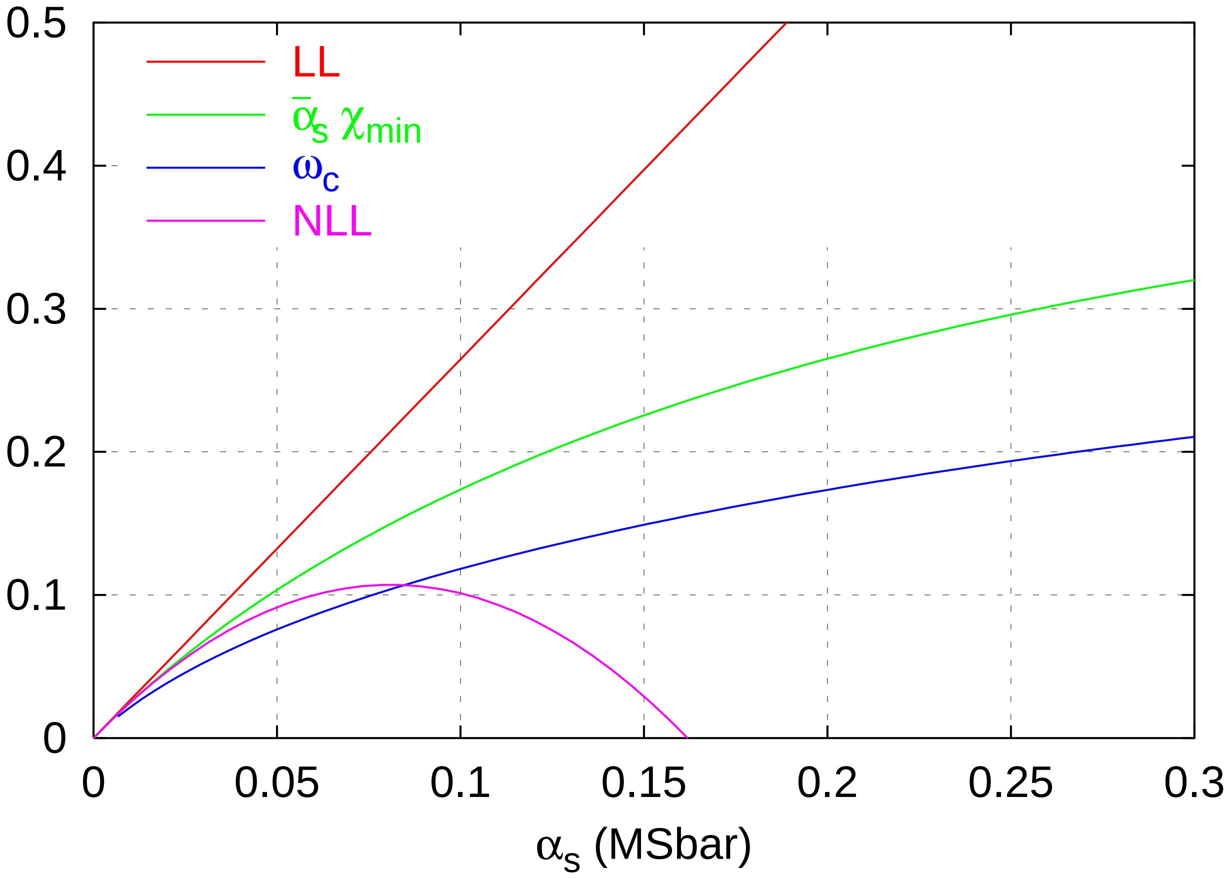
Because of strong suppression, phenomenology is much more promising (for a first, approximate, attempt, see Thorne 1999).

# Anomalous dimension power

Power of splitting functions differs from pure LL+NLL result:

$$\omega_c = \bar{\alpha}_s \chi_{\min} - 11.2\bar{\alpha}_s^{5/3} + \dots$$

Series of non-integer powers of $\alpha_s$ is also resumable.



# Conclusions

NLL corrections seem to lead to nonsensical predictions.

A major part of the NLL terms has a clear (collinear) origin $\Rightarrow$ this motivates a collinear resummation, which gives stable results.

At the qualitative level, perturbative BFKL is no longer in blatant contradiction with the data:

- resummed power in accord with (rather unprecise) experimental data
- small-$x$ anomalous dimensions are much suppressed compared to LL BFKL

Our understanding needs to be carried over to precise phenomenology (with the 'detail' that $\gamma^*$ impact factors remain to be calculated to NLO).

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From a more theoretical viewpoint: we are gradually improving our understanding of interaction between PT and NP effects (tunneling, factorisation).

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For the future? Unitarity. Soft pomeron...!